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NORMAL FUNCTORS, POWER SERIES AND λ-CALCULUS

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Introduction

This paper presents a new approach to the general subject known as 'continuous functionals' which includes computer-science oriented topics such as models of various λ-calculi.

Let us describe shortly the main features of this approach: a *type* is a collection of atomic 'states'; if A is a type, an *object of type A* is any superposition of atomic states. So an object of type A can be viewed as a sum $\sum_{a\in A} m_a \cdot \boldsymbol{a}$, where the 'vectors' $\boldsymbol{a}$ represent the atomic states, and the m_a's are cardinal numbers (in good cases integers) counting the multiplicities of each atomic state. So this differs from current constructions in two ways:

(i) because we count multiplicities (instead of just considering whether a state is present),

(ii) because no inner compatibility between the states is required: all combinations are accepted.

Let us say shortly that for the usual types of data, this makes a reasonable sense: certain combinations represent the usual outputs of computations, and general combinations are just formal superpositions of these usual outputs: such superpositions are made natural to consider if we think of parallel programs (then orders of multiplicity count the number of ways of getting the answer: typically in the case of the parallel 'or', if we have an atomic state T for 'true', then $\boldsymbol{T}$ or $\boldsymbol{T} = 2 \cdot \boldsymbol{T}$, because there are two ways to compute the value...), but also of random programs, where we can consider the superposition of all possible outputs depending on the random choices made during the execution. Section 4 which is quite poor in results, gives some idea how to interpret recursive functions in that framework.

The origin of the work was in the need for some link between say functionals of finite type and Π^1_2-logic. The results of [5, Ch. XII] are certainly of interest in this connection (see for instance the remarkable work of Päppinghaus [7]) but they do not interpret the equations in the standard way (the recursion equation is modified to make it increasing).

In this area, there are already well-known notions, such as Scott domains, Ershov f-spaces etc.; hence the problem was the compatibility of a functorial approach in the style of Π^1_2-logic with these notions. In fact it is well-known that the topological spaces considered in these works are extremely uneven ones (for instance a binary function is continuous iff separately continuous), so it is permitted to think that topology is not perhaps the standard way of attacking these problems; moreover categories have the advantage of containing a continuity notion (direct limits) but also additional kinds of limits that are not of any simple meaning in topology. So the conclusion was to abandon all topological features in favor of category-theoretic ones.

Let us see what this adds to the question of the function space: topological spaces have been replaced by categories, so continuous functions will be replaced by functors preserving a certain number of limits: direct limits, infinite pull-backs and kernels. (This functorial approach is superior to the topological one even in very simple cases, namely partially ordered sets; a poset can be viewed as a category, and then when X and Y are posets, a functor of the associated categories will be simply an increasing map; preservation of direct limits will be continuity w.r.t. supremum; but preservation of pull-backs will introduce a new idea: minimum data property. In Appendix A, we develop a notion of 'qualitative domain' which is basically a simplification of Scott domains, made possible by the minimum data property. The minimum data property and the particular ordering of functions had already been considered by Berry [1]; the fact that these properties were later found again from abstract considerations about pull-backs, is an evidence for their intrinsic interest. The concept of a qualitative domain is new.)

The functors involved in the function spaces are called *normal* functors. Now it is possible to prove a *normal form theorem* for normal functors, reminiscent of the normal form theorem for dilators (see [4]; the normal form theorem for dilators is a generalization of the Cantor Normal Form theorem). The theorem is not so easy, and uses all the preservation properties. Now, by counting the normal forms, it is possible to express normal functors (up to isomorphism) by power series: in other terms normal functors are analytic (and conversely). If we consider the normal functors from SET^A, to SET^B, then by looking at the coefficients of their power series expansion, we see that such functors can be viewed as the objects of a new category $\mathrm{SET}^{\mathrm{Int}(A)\cdot B}$. All usual operations (typically: λ-abstraction, application of a function to an argument) can be represented by means of normal functors, i.e., power series. It may be of interest to say a word about the meaning of a basic monomial, for instance $m_a^2 \cdot m_b^3$: such a monomial contributes to some multiplicity, and is expressed from the multiplicities m_a and m_b; concretely this means that the input 'a' has been used twice while the input 'b' has been used three times... In particular, linear functors are those where informations are used once; for instance application of a function to an argument is linear in the function (not at all in the argument!), similarly projections are linear. It seems that the general idea of an 'evaluation operation' is well represented by the idea of a linear functor: it is by following this principle that we arrive at the interpretation of disjunctive types in Appendix B: this interpretation is very different from the usual ones, and satisfies all imaginable requirements.

λ-calculus is easy to interpret: it suffices to solve the equation $\mathrm{Int}(A) \cdot A = A$ which is quite an obvious thing. It can be shown that the interpretation of normalizable terms are always with finite coefficients (i.e., multiplicities). This makes an essential use of asymmetric interpretation of normal proofs (three-valued semantics of cut-free proofs) familiar from work in proof-theory, see [3]. In fact it is possible to chop the models in many slices that measure the degree of finiteness of the objects; very little is known about these slices. One will also find in the text a very simple proof (under simple assumptions) that the fixed point of the syntax coincides with the smallest fixed point.

The system of Gödel, $\mathcal{T}$, is easily interpreted too; one has only to define the atomic states corresponding to integers in such a way that the primitive recursion will be interpreted: here the atomic states will be of the form N_k (equal to k) and U_k (strictly greater than k); without the U_k's the equations do not hold with variables. (The U_k's are also very useful for the interpretation of the μ-operator, see Section 4.) All the interpretations have finite coefficients.

It would have been the place then to consider systems with variable types, typically the system $\mathcal{F}$ introduced by the author in [2], and which contains second-order types. This requires some modification of the original pattern, and so this makes another paper. This has been done by the author in [9] using qualitative domains. More recently A. Martino did it with power series.

Section 7 finds its origin in a discussion with Per Martin-Löf: he expressed the idea of interpreting a random algorithm as a measure on some space (e.g., a Scott domain). The question solved here is just to find which is the right way of encoding the measure so that the operations between such measures become effective.

Finally let us mention a paper where ideas of counting multiplicities (in the framework of modules) appeared: this is the work of Main & Benson [6], which has at least some superficial analogy with what we present here.

Added in print. It appears now (October 1986) that the main interest of the paper is the general analogy with linear algebra. The analogy brought in sight new operations, new connectives, thus leading to 'linear logic'. The treatment of the sum of types (Appendix B) contains implicitly all the operations of linear logic. What has been found later is that the operations used here (e.g., linearization by means of 'tensor algebra') are of logical nature.

1. The category SET

1.1. Definition. The category SET of sets is defined by:

objects: sets, *morphisms*: functions.

Set will be the class of objects of SET, whereas SET(x, y) will denote, as expected the set of all morphisms from x to y. It will be often convenient to use the notation y^x.

1.2. Definition. Let A be a set; the category SET^A is defined as the product of 'A copies' of SET. We can in fact use the following representation for objects and morphisms of SET^A: the *objects* are formal sums $\sum_{a\in A} x_a \cdot \boldsymbol{a}$ with $x_a \in$ Set for all $a \in A$; the *morphisms* from $\sum_{a\in A} x_a \cdot \boldsymbol{a}$ to $\sum_{a\in A} y_a \cdot \boldsymbol{a}$ are formal sums $\sum_{a\in A} f_a \cdot \boldsymbol{a}$, with $f_a \in$ SET(x_a, y_a) for all $a \in A$.

Consistent with our notations of 1.1., Set^A will denote the class of objects of SET^A, whereas SET$^A(\boldsymbol{u}, \boldsymbol{v})$ will denote the set of all morphisms from $\boldsymbol{u}$ to $\boldsymbol{v}$. By the way, we shall often use the vectorial notation to denote objects and morphisms in SET^A. In particular, $\boldsymbol{a}$ will denote the object $\sum_{b\in A} x_b \cdot \boldsymbol{b}$, with $x_b = 0$ (i.e. $= \emptyset$) except for $b = a$, in which case $x_b = 1$ $(= \{\emptyset\})$.

1.3. Definition. (i) An object of SET is *finite* if it is finite as a set; it is an *integer* if as a set it is an integer, i.e., a finite ordinal.

(ii) An object $\boldsymbol{u} = \sum x_a \cdot \boldsymbol{a}$ of SET^A is *finite* iff the sum is of finite support (i.e., all x_a's but a finite number are zero) and all coefficients x_a are finite. Moreover, if all coefficients x_a are integers, then $\boldsymbol{u}$ is said to be *integral*. Int(A) denotes the set of all integral objects of SET^A.

1.4. Proposition. (i) *In* SET^A *all direct systems admit direct limits.*

(ii) *Let* (u_i, f_{ij}) *be a direct system in* SET *and let* (y, f_i) *be a candidate for its direct limit, i.e.,* $f_i \in$ SET(u_i, u), $f_j f_{ij} = f_i$. *Then* $(u, f_i) = \varinjlim(u_i, f_{ij})$ *iff the two following conditions are fulfilled*:

(Union property) $u = \bigcup_i \mathrm{rg}(f_i)$.

(Equality property) *If* $z_i, z_i' \in u_i$ *are such that* $f_i(z_i) = f_i(z_i')$, *then for some* $j \geqslant i$ *one has*: $f_{ij}(z_i) = f_{ij}(z_i')$.

(iii) *Let* $(\boldsymbol{u}_i, \boldsymbol{f}_{ij})$ *be a direct system in* SET^A, *with* $\boldsymbol{u}_i = \sum_a u_i^a \cdot \boldsymbol{a}$, $\boldsymbol{f}_{ij} = \sum_a f_{ij}^a \cdot \boldsymbol{a}$. *Then* $\varinjlim(\boldsymbol{u}_i, \boldsymbol{f}_{ij}) = (\sum_a u^a \cdot \boldsymbol{a}, \sum_a f_i^a \cdot \boldsymbol{a})$ *where* u^a *and* f_i^a *are defined by*

$$(u^a, f_i^a) = \varinjlim(u_i^a, f_{ij}^a).$$

Proof. Let us first recall that we mean by direct system a directed inductive system, i.e., that the index set I is directed. Also observe that, although we speak of 'the' direct limit, this limit is unique only up to isomorphism: if $\boldsymbol{h}$ is an isomorphism from $\boldsymbol{u}$ to $\boldsymbol{v}$, then $(\boldsymbol{v}, \boldsymbol{hf}_i)$ is also 'the' direct limit of $(\boldsymbol{u}_i, \boldsymbol{f}_{ij})$, and conversely any direct limit for (u_i, f_{ij}) can be expressed (given a direct limit (u, f_i)) in this way, with a unique h.

Property (iii) of the proposition is immediate: direct limits can be computed on each co-ordinate; by the way this reduces (i) to the case $A = 1$. In order to compute the direct limit of (u_i, f_{ij}) in SET, we consider the disjoint sum of the u_i's, i.e., the set $U = \{(i, z); z \in u_i\}$. On U we define an equivalence relation R by $(i, z) \equiv (j, z')$ iff for some $k \geqslant i, j$: $f_{ik}(z) = f_{jk}(z)$. Then we let $u = U/R$ and $f_i(z) = \dot{z}$, the equivalence class of z modulo R. We leave the details of checking that (u, f_i) is the direct limit of (u_i, f_{ij}) and similarly, the straightforward characterization of (u, f_i) by means of the pair of conditions of (ii). □

1.5. Theorem. *In* SET^A, *every object can be expressed as the direct limit of a system* (d_i, f_{ij}), *with* $d_i \in \mathrm{Int}(A)$.

Proof. Let $u = \sum_a u_a \cdot a \in \mathrm{Set}^A$; we define the index set I to consist of all finite sums $i = i_a \cdot a$ of finite subsets of u_a, $i_a \subset u_a$. We order I by $\sum_a i_a \leqslant \sum_a j_a$ iff $i_a \subset j_a$ for all a. When $i \leqslant j$, we define f^{ij} from i to j to be the sum $\sum_a f_a^{ij} \cdot a$ of the inclusion maps from the i_a's to the j_a's. Furthermore we can define f^i from i to u to be the sum $\sum_a f_a^i \cdot a$, where f_a^i is the inclusion map from i_a to u_a. Then it is plain that $(u, f^i) = \varinjlim(i, f^{ij})$, so u is the direct limit of a direct system of finite objects. Of course these finite objects can be replaced by isomorphic ones, so u is also the direct limit of a direct system of integral objects. □

1.6. Definition. (i) A morphism $f \in \mathrm{SET}^A(u, v)$ is *injective* iff all its coefficients f_a are injective functions.

(ii) A morphism $f \in \mathrm{SET}^A(u, v)$ is *surjective* if all its coefficients f_a are surjective functions.

(We have preferred to keep the more expressive terminology 'injective, surjective' instead of 'mono, epi' morphisms: keep in mind that the categories we are dealing with are very easy to understand, so abstract considerations are not always very useful...)

1.7. Proposition. *Assume that I is a non-empty set (N.B.: in some applications we need I to be infinite), and that $f^i \in \mathrm{SET}(x^i, y)$ for $i \in I$. Then (x, g^i) is the pull-back of the family f^i of morphisms (notation: $(x, g^i) = \bigwedge_i f^i$) iff the following holds:*

(i) $g^i \in \mathrm{SET}(x, x^i)$ *for all i.*

(ii) $f^i g^i = f^j g^j$ *for all $i, j \in I$.*

(iii) *Given any family (z^i), $z^i \in x^i$ such that $f^i(z^i) = f^j(z^j)$ for all $i, j \in I$, one can find a unique $z \in x$ such that $z^i = g^i(z)$ for all $i \in I$.*

Proof. The proposition is folkloric; it can be taken as the definition of the pull-back in the category SET. □

1.8. Proposition. (i) *In* SET^A *we have*

$$\left(\sum_a u_a \cdot a, \sum_a g^i_a \cdot a\right) = \bigwedge_i \left(\sum_a f^i_a \cdot a\right) \quad \textit{iff for all } a \in A \quad (u_a, g^i_a) = \bigwedge_i f^i_a.$$

(ii) *In* SET^A *pull-backs always exist.*

Proof. (i) expresses that pull-backs can be computed separately on each coordinate. This is immediate if one has in mind the category-theoretic definition of pull-backs. (ii) can be reduced, in view of (i), to the case $A = 1$, i.e., to the case of SET. In order to construct the pull-back of the f^i's ($f_i \in \mathrm{SET}(x^i, y)$) simply take the product X of all x^i's. In X, consider the subset x formed of all sequences $(z^i)_{i\in I}$ such that $f^i(z^i) = f^j(z^j)$ for all $i, j \in I$. The functions $g^i \in \mathrm{SET}(x, x^i)$ are defined by $g^i((z^j)_{j\in I}) = z^i$. It is plain that conditions (i)–(iii) of Proposition 1.7. are fulfilled, so (x, g^i) is 'the' pull-back of the f^i's. □

1.9. Remarks. (i) Pull-backs should not be confused with intersections; for instance, if $f \in \mathrm{SET}(3, 1)$, $g \in \mathrm{SET}(2, 1)$, then the pull-back of f, g is $(6, (h, k))$, with $h \in \mathrm{SET}(6, 3)$, $k \in \mathrm{SET}(6, 2)$, defined by:

$$h(0) = h(3) = 0, \qquad h(1) = h(4) = 1, \qquad h(2) = h(5) = 2,$$
$$k(0) = k(2) = k(4) = 0, \qquad k(1) = k(3) = k(5) = 1.$$

(ii) This shows that pull-backs are more complicated than in categories like ON (see [4]); this is because, in ON, all morphisms are injective functions. In fact pull-backs have a lot to do with injectivity: in SET^A, injectivity of morphisms can be characterized by means of pull-backs. To say that $f \in \mathrm{SET}^A(u, v)$ is injective is equivalent to say that the diagram

$$\begin{tikzcd} u \arrow[r, "i_u"] \arrow[d, "i_u"'] & u \arrow[d, "f"] \\ u \arrow[r, "f"'] & v \end{tikzcd}$$

is cartesian, i.e., is a pull-back diagram (with $I = 2$, $x^0 = x^1 = x = u$, $y = v$, $f^0 = f^1 = f$, $g^0 = g^1 = i_u$ (the identity of u)).

As a consequence, as soon as a functor preserves pull-backs, then the image of an injective morphism will be an injective morphism.

1.10. Proposition. *Assume that* $f, g \in \mathrm{SET}(y, z)$. *Then the kernel* $\mathrm{Ker}(f, g)$ *of* f *and* g *is characterized by the following conditions*:

(i) $\mathrm{Ker}(f, g)$ *is a morphism whose target is* y; *let* x *be such that* $\mathrm{Ker}(f, g) \in \mathrm{SET}(x, y)$.

(ii) $\mathrm{Ker}(f, g)$ *is injective.*

(iii) *The range of the function* $\mathrm{Ker}(f, g)$ *is equal to the set* $\{t; f(t) = g(t)\}$.

Proof. Once more a folkloric result which can be taken as the definition of kernels in SET. □

1.11. Proposition. (i) *If* $f = \Sigma_a f_a \cdot a$ *and* $g = \Sigma_a g_a \cdot a$, *then*

$$\mathrm{Ker}(f, g) = \sum_a \mathrm{Ker}(f_a, g_a) \cdot a.$$

(ii) *In* SET^A, *kernels always exist.*

Proof. (i) is a triviality from the category-theoretic definition of kernels; this means that the kernels can be computed separately on each coordinate.

(ii) can be reduced to the case $A = 1$, i.e., to the case of SET: take x to be the set $\{t; f(t) = g(t)\}$ and let $\mathrm{Ker}(f, g)$ be the inclusion map from x to y. □

1.12. Remarks. (i) The equality of two morphisms can be expressed in terms of kernels: $f = g$ (if $f, g \in \mathrm{SET}^A(y, z)$) iff $\mathrm{Ker}(f, g)$ is surjective.

(ii) If $h \in \mathrm{SET}(x, y)$ is injective, then h can be expressed as $\mathrm{Ker}(f, g)$ for some well-chosen f, g and some well-chosen z such that $f, g \in \mathrm{SET}^A(y, z)$. As a corollary, a functor preserving kernels will preserve injectivity of morphisms. Hence normal functors will have two reasons to preserve injectivity of morphisms: preservation of pull-backs and preservation of kernels.

(iii) All these definitions will be used in Section 2 to define the concept of a normal functor, preserving direct limits, kernels and pull-backs. Typical examples of such functors are given by the sum and the product (both binary, see below for their precise definition). Our claim is that, to some extent, there is no other normal functor, i.e., that up to isomorphism, a normal functor can be expressed by means of sums and products. This will be proved in Section 2, by showing that "any normal functor is analytic". An analytic functor is a functor expressed by means of sums of monomials

$$c_{n_1,a_1,\ldots,n_k,a_k;b} \cdot x_{a_1}^{n_1} \cdots x_{a_k}^{n_k} \cdot b$$

The coefficients $c_{\ldots}$ will define the functor up to isomorphism; now the product $x_{a_1}^{n_1} \cdots x_{a_k}^{n_k}$ is equal (or isomorphic) to the set of all morphisms in SET^A from $n_1 \cdot a_1 + \cdots + n_k \cdot a_k$ to $x_1 \cdot a_1 + \cdots + x_k \cdot a_k$. This is why expressions $\mathrm{SET}^A(d, x)$ occur in the notations for normal functors. A last word about the relation between 'normal' and 'analytic': the analyticity of normal functors will be established by means of a normal form theorem, i.e., a unique representation theorem involving morphisms as parameters; by counting the number of possible normal forms for a given functor and a given input, we shall essentially find a sum of expressions $\mathrm{SET}(d, x)$, and this is how the theorem works. We now introduce the notations for Section 2:

1.13. Notations. We shall denote i_x the identity of x in SET; in SET^A the identity of u will be denoted i_u. However, in some cases, we shall simply note these identity morphisms x, u, for instance, in power series expansions $\sum_d C_{d,b} \cdot \mathrm{SET}^A(d, f)$, which denotes the product (see below) of a set $C_{d,b}$ with a function $\mathrm{SET}^A(d, f)$; of course, $C_{d,b}$ should be replaced by the identify of $C_{d,b}$. By the way, let us introduce $\mathrm{SET}^A(d, f)$: this is a function from $\mathrm{SET}^A(d, u)$ to $\mathrm{SET}^A(d, v)$ defined, when $f \in \mathrm{SET}^A(u, v)$, by $\mathrm{SET}^A(d, f)(h) = fh$.

Let us close these remarks on the notations for morphisms by saying that very often in the definition of a functor the part of the definition concerning morphisms will be omitted, because it is just like the definition for objects, for instance in

$$F_b(u) = \sum_d C_{d,b} \cdot \mathrm{SET}(d, u) \quad \text{etc.}$$

The 'etc.' means that the definition on morphisms is $F_b(f) = \sum_d C_{d,b} \cdot \mathrm{SET}(d, f)$.

1.14. Definition. (i) The functor sum from SET × SET to SET is defined by:

$$x + y = \{(i, t); (i = 1 \text{ and } t \in x) \text{ or } (i = 2 \text{ and } t \in y)\}.$$

If $f \in \mathrm{SET}(x, x')$, $g \in \mathrm{SET}(y, y')$, then $f + g \in \mathrm{SET}(x + y, x' + y')$ is defined by

$$(f + g)((1, t)) = (1, f(t)), \qquad (f + g)((2, t)) = (2, g(t)).$$

(ii) The functor sum from $\mathrm{SET}^A \times \mathrm{SET}^A$ to SET^A is defined by

$$\Big(\sum_a x_a \cdot a\Big) + \Big(\sum_a y_a \cdot a\Big) = \sum_a (x_a + y_a) \cdot a \quad \text{etc.}$$

1.15. Remark. This should be now the place of listing properties of the sum: the sum is commutative, associative, 0 is neutral etc. However these properties are not literally true, but only up to isomorphism: for instance there is an isomorphism (bijective natural transformation) between the functors $F(u, v) = u + v$ etc. and the functor $G(u, v) = v + u$ etc. There is no harm in identifying two isomorphic sets provided 'functoriality' is preserved, i.e., that this is compatible with the action of morphisms. In practice, we shall often consider that the sets to which one applies the sum are disjoint, so the sum will simply be a union.

1.16. Definition. (i) We define the functor product from SET × SET to SET by:

$$x \cdot y = \{(t, u); t \in x \text{ and } u \in y\}, \qquad (f \cdot g)((t, u)) = (f(t), g(u)).$$

(ii) We define the functor product from $\mathrm{SET} \times \mathrm{SET}^A$ to SET^A by

$$x \cdot \sum_a y_a \cdot a = \sum_a (x \cdot y_a) \cdot a \quad \text{etc.}$$

1.17. Remark. It would now be the place to state the obvious associativity and distributivity properties of the product; the problem is the same as with the sum: this is true up to isomorphism. In practice we shall use the sum and the product as freely as we would in a vector space, although the basic 'field' is SET!

1.18. Definition. An object of SET^A is said to be *recursive* when it is the direct limit of a recursive direct system (d_i, f_{ij}) with the d_i's in $\mathrm{Int}(A)$. In fact this implicitely implies that

– A is effectively enumerated (in practice this is always the case).

– The index set I is effectively enumerated, and the order of I is effective; in practice, one can always take $I = \mathbb{N}$.

Recursive objects are closed under all reasonable operations. In the sequel, we shall rarely mention the recursivity of the objects, because this is most of the time obvious. The notion of recursivity can be transferred to normal functors: normal functors can be represented by objects of $\mathrm{SET}^{\mathrm{Int}(A)\cdot B}$ etc.

2. Normal and analytic functors

2.1. Definition. Let F be a functor from SET^A to SET^B; F is said to be *normal* if it commutes with the following limits:

(i) direct limits, i.e., directed inductive limits,

(ii) pull-backs (including infinite pull-backs),

(iii) kernels.

2.2. Definition. Let F be a functor from SET^A to SET^B. F is said to be *analytic* if one can find a family $(C_d)_{d\in\mathrm{Int}(A)}$ of objects of SET^B such that F is the functor:

$$F(u) = \sum_{d\in\mathrm{Int}(A)} \mathrm{SET}^A(d, u)\cdot C_d, \qquad F(f) = \sum_{d\in\mathrm{Int}(A)} \mathrm{SET}^A(d, f)\cdot C_d.$$

2.3. Theorem. *Let F be a functor from* SET^A *to* SET^B. *Then F is normal iff F is isomorphic to an analytic functor from* SET^A *to* SET^B.

Proof. This theorem will indeed be the main result of this section, and we postpone its proof for a while; see 2.8.

2.4. Comments. (i) The term 'normal' comes from the fact that normal functors enjoy a kind of 'normal form theorem', analogous to the Cantor Normal Form theorem and to the Normal Form Theorem for dilators [4].

(ii) The term 'analytic' comes from the fact that an analytic functor has really a 'power series expansion', analogous to the power series which arise in complex analysis. Of course no deep connection with complex analysis can be reasonably expected!

2.5. Notations. When F is analytic, we can also use the notation

$$F(u) = \sum_{d \in \mathrm{Int}(A)} u^d \cdot C_d, \qquad F(f) = \sum_{\cdot d \in \mathrm{Int}(A)} f^d \cdot C_d$$

which stresses the analyticity. If A where linearily ordered, then it would even be possible to replace, when $u = \sum_a u_a \cdot a$ and $d = n_1 \cdot a_1 + \cdots + n_k \cdot a_k$ with $n_1, \ldots, n_k \neq 0$ and $a_1 > \cdots > a_k$, the 'monomial' u^d by the other monomial $u_{a_1}^{n_1} \cdots u_{a_k}^{n_k}$ etc.

Another possibility will be to consider the functors F_b from SET^A to SET such that: $F(u) = \sum_{b \in B} F_b(u) \cdot b$ etc. The functors F_b are analytic: if $C_d = \sum_{b \in B} C_{d,b} \cdot b$ then it is plain that

$$F_b(u) = \sum_{d \in \mathrm{Int}(A)} u^d \cdot C_{d,b} \quad \text{etc.}$$

It will turn out later that the family $(C_{d,b})$ indexed by $\mathrm{Int}(A) \cdot B$ plays an essential role.

2.6. Definition. Let F be a functor from SET^A to SET. If $u \in \mathrm{Set}^A$ and $z \in F(u)$, then a *form* for z (w.r.t. F and u) is any 3-tuple (z^*, d, f) such that:

(i) $d \in \mathrm{Set}^A$, (iii) $z^* \in F(d)$,

(ii) $f \in \mathrm{SET}^A(d, u)$, (iv) $z = F(f)(z^*)$.

We use the notation $z = |z^*, d; f; u|_F$ to summarize these facts.

2.7. Definition. Let $z = |z^*; d; f; u|_F$ be a form. Then

(i) The form is said to be *normal* when it enjoys the following universal property: given any other form $z = |y^*; e; g; u|_F$ (same F and u) there is a *unique* $h \in \mathrm{SET}^A(d, e)$ such that $y^* = F(h)(z^*)$, i.e., $y^* = |z^*; d; h; e|_F$. Normal forms are represented by $(\cdots)$ instead of $|\cdots|$: $z = (z^*; d; f; u)$.

(ii) The form is said to be *saturated* if given any form $z^* = |z^{**}; e; g; d|_F$, then the morphism g is surjective.

(iii) When d is finite, the form is said to be *finite*; when d is integral, the form is said to be *integral*; when f is an isomorphism the form is said to be *trivial.*

2.8. Theorem. *Let F be a functor from* SET^A *to* SET. *The following are equivalent*:

(i) *F is normal.*

(ii) *F is isomorphic to a normal functor.*

(iii) (Normal form property) *Given any $u \in \mathrm{Set}^A$ and any $z \in F(u)$, then z has a finite normal form w.r.t. F and u, i.e., $z = (z^*; d; f; u)_F$ with d finite.*

Proof. The proof of the theorem is postponed; let us remark that 2.8 implies 2.3: if we write $F(u) = \sum_{b \in B} F_b(u) \cdot b$ etc., then F is normal iff all F_b's are normal, F is analytic iff all F_b's are analytic.

2.9 Examples. We consider the functor $F(x)=x^2$ $(=x\cdot x)$, $F(f)=f^2$ $(=f\cdot f)$ from SET to SET and we consider the point $(4,4)$ in F(23).

(i) Let f be any function from 7 to 23 such that $f(3)=4$; then $(4,4)=|(3,3);7;f;23|_F$. This form has no special property.

(ii) Let g be any function from 6 to 23 such that $g(1)=g(5)=4$; then $(4,4)=|(1,5);6;g;23|_F=|(5,1);6;g;23|_F$. These forms have no special property.

(iii) Let h be the function from 1 to 23 defined by $h(0)=4$; then $(4,4)=|(0,0);1;h;23|_F$. This form is saturated. But the form is not normal: if we use example (ii) above, it is not possible to write $(1,5)$ as $|(0,0);1;h';6|_F$. But the normality condition is verified on example (i).

(iv) Let k be the function from 2 to 23 defined by $k(0)=k(1)=4$; then $(4,4)=|(0,1);2;k;23|_F$. This form is both normal and saturated. The same is true of the symmetric form $(4,4)=|(1,0);2;k;23|_F$. As a matter of fact, normal forms are always saturated, but we already know that saturated forms may be non-normal.

2.10. Proposition. Let $z=(z^*,d;f;u)_F$ be a normal form. Then

(i) *If $z=|z^*;d;g;u|_F$ for some g, then $f=g$.*

(ii) *If $z=|y^*;e;g;u|_F$ let h be the unique solution of $y^*=|z^*;d;h;e|_F$. Then $f=gh$. Moreover $y^*=(z^*;d;h;e)_F$.*

(iii) *If $z=(y^*;e;g;u)_F$, then there is a unique isomorphism h from d to e such that $y^*=F(h)(z^*)$. Moreover $f=gh$.*

Proof. (i) Consider the trivial form $z=|z;u;i_u;u|_F$: by universality of the normal form, the equation $z=|z^*;d;g;u|_F$ has only one solution in g, namely $g=f$.

(ii) By 'composition of forms', we obtain $z=|z^*;d;gh;u|_F$. So by (i) we obtain $gh=f$. It remains to prove that the form $|z^*;d;h;e|_F$ is normal: but if $y^*=|y^{**};e';h';e|_F$, then $z=|y^{**};e';gh';u|_F$ and so the equation $y^{**}=|z^*;d;k;e'|_F$ has exactly one solution in k.

(iii) is left to the reader. □

2.11. Proposition. *Let F be a functor from* SET^A *to* SET *enjoying the normal form property* 2.8(iii). *Then F is isomorphic to an analytic functor.*

Proof. Let us see the way F acts on normal forms: we claim that, when $f\in\mathrm{SET}^A(u,v)$, $F(f)((z^*;d;h;u)_F)=(z^*;d;fh;v)_F$. (*Proof.* One has to show the normality of the form $|z^*;d;fh;v|_F$. Since F enjoys the normal form property, $|z^*;d;fh;v|_F=(z^{**};e;fhk;v)_F$ for some z^{**},e,k, and $z^*=F(k)(z^{**})$. By 2.10(ii) the form $|z^{**};e;hk;u|_F$ is normal, so by 2.10(iii) applied to the normal forms $(z^{**};e;hk;u)_F$ and $(z^*;d;h;u)_F$, k is an isomorphism, hence the form $|z^*;d;fh;v|_F$ is normal. □) Now, for each $d\in\mathrm{Int}(A)$, consider the set X_d of all points $z \in F(d)$ such that the form $|z; d; i_d; d|_F$ is normal. On X_d we define an equivalence relation R_d by

$$z \equiv z; (\bmod R_d) \quad \text{iff} \quad z = F(f)(z') \quad \text{for some automorphism } f \text{ of } d.$$

For each equivalence class modulo R_d, we select an element in the class, and we define C_d to be the set of all selected points. Then we get a unique normal form theorem: given $u \in \mathrm{Set}^A$ and $z \in F(u)$, there are unique $d \in \mathrm{Int}(A)$, $z^* \in C_d$ and $f \in \mathrm{SET}(d, u)$ such that $z = (z^*; d; f; u)$. (*Proof.* z has a finite normal form $(y^*; e; h; u)_F$; using an isomorphism k from some integral object d to e, we can obtain an integral normal form $(y^{**}; d; hg; u)_F$ for z. Now, by 2.10(ii) $y^{**} \in X_d$, so there is a unique point $z^* \in C_d$ such that $y^{**} \equiv z^* \bmod R_d$. Let k be an automorphism of d such that $y^{**} = F(k)(z^*)$; then $z = (z^*; d; f; u)_F$ with $f = hgk$. The unicity of the solution is a clear consequence of 2.10(iii). □) In other terms, $F(u)$ can be put in bijection with the set of all 3-tuples (d, f, z^*) such that $d \in \mathrm{Int}(A)$, $f \in \mathrm{SET}^A(d, u)$ and $z^* \in C_d$. This set is precisely the sum $\Sigma_{d \in \mathrm{Int}(A)} \mathrm{SET}^A(d, u) \cdot C_d$. Moreover, if h_u is this bijection, and if $h_u(z) = (d, f, z^*)$, then for any v and $g \in \mathrm{SET}^A(u, v)$: $h_v(F(g)(z)) = (d, gf, z^*)$. So, if we define an analytic functor G by

$$G(u) = \sum_{d \in \mathrm{Int}(A)} \mathrm{SET}^A(d, u) \cdot C_d, \qquad G(f) = \sum_{d \in \mathrm{Int}(A)} \mathrm{SET}^A(d, f) \cdot C_d,$$

it is plain that $h_v F(g) = G(g) h_u$, and since the h_u's are bijections, F is isomorphic to the analytic functor G. □

2.12. Proposition. *If F is an analytic functor from* SET^A *to* SET, *F enjoys the normal form property.*

Proof. Assume that $F(u) = \Sigma_{d \in \mathrm{Int}(A)} u^d \cdot C_d$ etc. Any point in $F(u)$ is a 3-tuple (d, f, z^*) with $z^* \in C_d$ and $f \in \mathrm{SET}^A(d, u)$. If $z_0 \in F(d)$ is defined by $z_0 = (d, i_d, z^*)$, then it is plain that (d, f, z^*) has the form $|z_0; d; f; u|_F$ and that this form is finite, so it remains to prove that this form is normal: if $(d, f, z^*) = F(h)((e, k, y^*))$, this means that $d = e$, $f = hk$, $y^* = z^*$ so $(e, k, y^*) = |z_0; d; k; u|_F$, and k is obviously uniquely determined. □

2.13. Proposition. *Let F be a functor from* SET^A *to* SET *enjoying the normal form property. Then F is normal.*

Proof. First observe that F must preserve injectivity of morphisms: if $f \in \mathrm{SET}^A(u, v)$ is injective, then let $z = (z^*; d; g; u)_F$ and $y = (y^*; e; h; u)_F$ be two points in $F(u)$; if $F(f)(z) = F(f)(y)$, this means that $(z^*; d; fg; v)_F = (y^*; e; fh; v)_F$, so there is an isomorphism k such that $z^* = F(k)(y^*)$, $fh = fgk$; since f is injective, $h = gk$, so $z = y$.

(i) *F preserves kernels.* Assume that $f, g \in \mathrm{SET}^A(v, w)$ and let $h \in \mathrm{SET}^A(u, v)$ be their kernel. One must show that $F(h) = \mathrm{Ker}(F(f), F(g))$. For this we must first check that $F(h)$ is injective, but we have just seen that F preserves injective morphisms. We must also show that $rg(F(h)) = \{z; F(f)(z) = F(g)(z)\}$ and in fact the only non-trivial point is the inclusion $\supset$. So let us take $z = (z^*; d; k; v)_F$ and assume that $F(f)(z) = F(g)(z)$; then by unicity properties of normal forms, $fk = gk$ and this forces the existence of a (unique) $k' \in \mathrm{SET}^A(d, u)$ such that $hk' = k$: but then $z \in \mathrm{rg}(F(h))$. So $F(h) = \mathrm{Ker}(F(f), F(h))$.

(ii) *F preserves direct limits.* Assume that $(u, f_i) = \varinjlim(u_i, f_{ij})$. If $z \in F(u)$ consider its normal form $(z^*; d; f; u)_F$. Since d is finite and the index set I is directed, it will be possible to find some index $i \in I$ and some $h \in \mathrm{SET}^A(d, u_i)$ such that $f = f_i h$. Then $F(f) = F(f_i)F(h)$, so $z \in \mathrm{rg}(F(f_i))$. This establishes the equality $F(u) = \bigcup_{i \in I} \mathrm{rg}(F(f_i))$, so the union property holds. Now assume that $z_i = (z_i^*; d; h; u_i)_F$, $y_i = (y_i^*; e; k; u_i)_F$ are such that $F(f_i)(z_i) = F(f_i)(y_i)$. Then without loss of generality we can assume that $d = e$, $z_i^* = y_i^*$ and then it turns out that $f_i h = f_i k$. Then it is easy to show the existence of some $j \geqslant_I i$ such that $f_{ij} h = f_{ij} k$. (Use the equality property for (u_i, f_{ij}), together with finiteness of d and directness of I.) Then $F(f_{ij})(z_i) = F(f_{ij})(y_i)$, so the equality property holds. We can conclude that $(F(u), F(f_i)) = \varinjlim(F(u_i), F(f_{ij}))$.

(iii) *F preserves pull-backs.* Assume that $f_i \in \mathrm{SET}^A(v_i, w)$, $g_i \in \mathrm{SET}^A(u, v_i)$ and $(g_i) = \bigwedge_{i \in I} f_i$, the indexing set being non-empty, possibly infinite. Take a point $z \in F(u)$; then $z = (z^*; d; h; u)_F$. If $i \in I$, consider $z_i = (z^*; d; g_i h; v_i)_F$ and then it is immediate that $F(f_i)(z_i) = F(f_j)(z_j)$ for all $i, j \in I$, so this proves that $(z_i)_{i \in I}$ is a point in the set x defined by the condition that x is the common source of the morphisms k_i such that $(k_i) = \bigwedge_{i \in I} F(f_i)$. So we have defined a function f from $F(u)$ to x. Conversely, given a point (z_i) in x, one has $F(f_i)(z_i) = F(f_j)(z_j)$ for all $i, j \in I$, and if one looks at the normal forms of the points z_i they must be of the form $z_i = (z^*; d; h_i; v_i)_F$ and furthermore $f_i h_i = f_j h_j$ for all $i, j \in I$. By definition of a pull-back, there is a morphism $h \in \mathrm{SET}^A(d, u)$ such that $h_i = g_i h$ for all $i \in I$. So we can associate to $(z_i) \in x$ the point $z = (z^*; d; h; u)_F$ of $F(u)$: this function is called g and it is plain that the functions f and g are reciprocal. Finally it is immediate that $F(g_i) = k_i f$, and this shows that $(F(g_i)) = \bigwedge_{i \in I} F(f_i)$. □

2.14. Theorem. *If F is a normal functor from* SET^A *to* SET, *then F enjoys the normal form property.*

Proof. The theorem results of a succession of lemmas.

2.14.1. Lemma. *All saturated forms w.r.t. F are finite.*

Proof. Let $|z^*; d; f; u|_F$ be a saturated form w.r.t. F; assume that $(d, f_i) = \varinjlim(d_i, f_{ij})$ with the d_i's finite. Then since F is normal, $(F(d), F(f_i)) =$

$\varinjlim((F(d_i), F(f_{ij})))$, and so there is some i such that $z^* \in \mathrm{rg}(F(f_i))$. But the form is saturated so f_i is surjective. Since d_i is finite, d is finite as well. □

2.14.2. Lemma. *If* $u \in \mathrm{Set}^A$, if $z \in F(u)$, *then* z *has a saturated form w.r.t.* F, u.

Proof. Let I be the set of finite subobjects of u, ordered by inclusion. To I we can associate the direct system (i, f_{ij}) whose direct limit is (u, f_i). It is possible to find $i \in I$ such that $z \in \mathrm{rg}(F(f_i))$. We can assume that i has been chosen minimal for this property, and we have a form $z = |z^*; i; f_i; u|_F$. This form is saturated, because if $z^* = |y^*; e; g; i|_F$ and g is not surjective, then we can assume w.l.o.g. that g is one of the inclusions f_{ji} with $j \subsetneq i$ (because g can be written $f_{ji}h$) and then $z \in \mathrm{rg}(F(f_j))$, contradicting the minimality of i. □

Now we can end the proof of 2.14: given $z \in F(u)$, consider the set of all integral saturated forms $|z_i^*; d_i; g_i; u|_F$. Let $(f_i) = \bigwedge_{i\in I} g_i$ and let e be the common source of the f_i's. Then since F is normal $(F(f_i)) = \bigwedge_{i\in I} F(g_i)$. This means the existence of $y^* \in F(e)$ such that $z_i = F(f_i)(y^*)$ for all i. Let $y^* = |z^*; d; h; e|_F$ be a saturated form for y^*. Then $|z^*; d; g_i f_i h; u|_F$ is a saturated form for z. It remains to prove that this form has the universal property of normal forms: so assume that $z = |x^*; c; g; u|_F$. By Lemma 2.14.2, x^* has a saturated form $|x^{**}; b; k; c|_F$ so $z = |x^{**}; b; gk; u|_F$ and so for some $i \in I$ $x^{**} = z_i$, $b = d_i$ (because by Lemma 2.14.1, b is finite, so one can assume that b is integral) and $gk = g_i$. Then $x^* = F((kf_i h)(z^*)$. The only missing point is the unicity of the morphism h' such that $x^* = F(h')(z^*)$. This comes from a general property of saturated forms w.r.t. a normal functor:

2.14.3. Lemma. *If the form* $z = |z^*; d; f; u|_F$ *is saturated, then the equation* $z = |z^*; d; g; u|_F$ *has exactly one solution in* g, *namely* $g = f$.

Proof. Take any solution of the equation $z = |z^*; d; g; u|_F$. Then if $h = \mathrm{Ker}(f, g)$ one must have $z^* \in \mathrm{rg}(\mathrm{Ker}(F(f), F(g))) = \mathrm{rg}(\mathrm{Ker}(F(h))$, and by definition of saturation, h must be surjective, which forces $f = g$. □

The lemma completes the proof of 2.14. □

Proof of 2.8. Put together 2.11, 2.12, 2.13 and 2.14. □

2.15. Remarks. (i) Saturated forms w.r.t. a normal functor F are exactly those forms enjoying the unicity condition of Lemma 2.14.3. More precisely, if the form $z^* = |z^*; d; i_d; d|_F$ is not saturated, then it is possible to find u and $z \in F(u)$ such that the equation $z = |z^*; d; g; u|_F$ has at least two solutions in g: if

$z^* \in \mathrm{rg}(F(h))$ with h not surjective, choose u and $f, g \in \mathrm{SET}^A(d, u)$ such that $fh = gh$ and $f \neq g$, and let $z = |z^*; d; f; u|_F = |z^*; d; g; u|_F$.

(ii) Now we give the precise description of all possible saturated forms w.r.t. a given normal functor. We start with a normal form $z = |z; d; i_d; d|_F$. Then given any surjective morphism with source d, say $f\, \mathrm{SET}^A(d, e)$, the point $z' = F(f)(z)$ is such that the form $|z'; e; i_e; e|_F$ is saturated. Of course, if one replaces f by hf, where h is an isomorphism from e to e', then we shall obtain an equivalent saturated form. In fact, if we are interested in generating only non-equivalent saturated forms, we can proceed as follows:

(1) Select an equivalence relation on d: this means, if $d = \sum_a n_a \cdot a$, that for all a, we have an equivalence relation R_a on n_a. We use the notation $R = \sum_a R_a \cdot a$ for such a 'relation'.

(2) If R is such an equivalence relation, we can construct $d/R = \sum_a (d_a/R_a) \cdot a$ and the canonical surjections S_R from d to d/R, defined by $S_R = \sum_a S_{R_a} \cdot a$.

(3) Then consider $z_R = (z; d; S_R; d/R)_F$, $e_R = d/R$. Then the pair (z_R, e_R) defines a saturated form w.r.t. F.

(4) The forms obtained in (3) are non-equivalent.

(5) All saturated forms w.r.t. F are equivalent to a saturated form obtained from a normal form via process (1)–(3) above.

(iii) Let us for instance consider the functor F from SET to SET: $F(x) = x^3$, $F(f) = f^3$. If $z^* = (0, 1, 2)$, then any point $z \in F(x)$ has a normal form $z = (z^*; 3; f; x)_F$. Now we describe all possible saturated forms:

(1) Forms $|(0, 1, 2); 3; f; x|_F$: this is the general normal form for F, and it corresponds to the equivalence relation with 3 classes.

(2) Forms $|(0, 1, 1); 2; f; x|_F$: the points enjoying such forms are all points (a, b, b). The corresponding equivalence relation has two classes and $1 \equiv 2$.

(3) Forms $|(1, 0, 1); 2; f; x|_F$: the points enjoying such forms are all points (b, a, b). The corresponding equivalence relation has two classes and $0 \equiv 2$.

(4) Forms $|(1, 1, 0); 2; f; x|_F$: the points enjoying such forms are all points (b, b, a). The corresponding equivalence relation has two classes and $0 \equiv 1$.

(5) Forms $|(0, 0, 0); 1; f; x|_F$: the points enjoying such forms are all points (a, a, a). The corresponding equivalence relation has one class: $0 \equiv 1 \equiv 2$.

(iv) Let us summarize in which way the various preservation properties are used to obtain normal form:

(1) Preservation of kernels entails a unicity property for saturated forms, namely Lemma 2.14.3.

(2) Preservation of direct limits entails the existence of saturated forms as well as finiteness of such forms: Lemmas 2.14.1 and 2.14.2.

(3) There is an obvious ordering between saturated forms for the same z, w.r.t. F and u; preservation of pull-backs shows that this (pre)ordering is completely directed, so there is a maximum saturated form, which is the finite normal form. If we only require preservation of finite pull-backs, then the set of

saturated forms will be directed, but may have no maximum element, because there may be infinitely many non-equivalent saturated forms.

2.16. Definition. The following data define the category $\mathrm{SET}^A \to \mathrm{SET}^B$:

Objects: normal functors from SET^A to SET^B.

Morphisms from F to G: the set of all cartesian natural transformations from F to G, i.e., the set of all families $(T_u)_{u \in \mathrm{Set}^A}$ such that:

(i) $T_u \in \mathrm{SET}^B(F(u), G(u))$ for all $u \in \mathrm{Set}^A$.

(ii) For all $u, v \in \mathrm{Set}^A$, for any $f \in \mathrm{Set}^A(u, v)$ the diagram is cartesian. This means that the diagram is commutative and furthermore that $(F(f), T_u) = T_v \mathbin{\&} G(f)$ (& indicates a binary pull-back).

We shall often use the notation $T(u)$ instead of Tu.

$$\begin{array}{ccc} F(u) & \xrightarrow{T_u} & G(u) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(v) & \xrightarrow[T_v]{} & G(v) \end{array}$$

2.17. Theorem. (i) *Assume that in* $\mathrm{SET}^A \to \mathrm{SET}$, *$T$ is a morphism from F to G. Then T sends normal forms to normal forms by means of the formula*:

$$T(u)((z^*; d; f; u)_F) = (T(d)(z^*); d; f; u)_G.$$

(ii) *The category* $\mathrm{SET}^A \to \mathrm{SET}^B$ *is closed under direct limits.*

Proof. (i) It is immediate to check that $|T(d)(z^*); d; f; u|_G$ is a form for $T(u)((z^*; d; f; u)_F)$. So we must prove that this form is normal. For instance the form is saturated: if $T(d)(z^*) = G(g)(y^*)$, then use the pull-back condition $(F(g), T(e)) = T(d) \mathbin{\&} G(g)$ $(g \in \mathrm{SET}^A(e, d))$, which entails the existence of $z^{**} \in F(e)$ such that $z^* = F(g)(z^{**})$ and $y^* = T(e)(z^{**})$. But the form $|z^*; d; f; u|_F$ is saturated, g is surjective, so the image form is saturated. If we use the fact that the original form is normal, then we get more: that g is an isomorphism, and this is enough to conclude that the image form is normal.

(ii) The property is easily reduced to its particular case $B = 1$. If (F_i, T_{ij}) is a direct system in $\mathrm{SET}^A \to \mathrm{SET}$, then the direct limit (if it exists) must be computed pointwise: $F(x) = \varinjlim(F_i(x), T_{ij}(x))$ etc. This clearly defines a functor F from SET^A to SET, and it will be enough to check the normality of F, which will follow from the normal form property for F. Now since the T_{ij}'s are cartesian, they preserve normal forms, and from this it is not very difficult to find the finite normal forms w.r.t. F. □

2.18. Remarks. (i) General natural transformations from F to G do not preserve normal forms, not even saturated forms. However, when the functions $T(u)$ are injective, T will send saturated forms on saturated forms.

(ii) Let us give the example of a functor from SET to SET which preserves direct limits, kernels, finite pull-backs, but which is not normal: define $F_n(x) = x^{n+1}$, $F_n(f) = f^{n+1}$; when $n \leq m$, define a (non-cartesian) natural transformation T_{nm} from F_n to F_m by

$$T_{nm}(x)(a_0, \ldots, a_n) = (a_0, \ldots, a_{n-1}, a_n, \ldots, a_n).$$

The pointwise direct limit F of the system (F_n, T_{nm}) preserves all kinds of limits we are interested in, but infinite pull-backs. This functor has no normal form property. In fact $F(x)$ can be viewed as the set of all infinite sequences $(a_0, \ldots, a_n, \ldots)$ of points of x that are eventually constant.

This example shows the importance of the cartesianity of the natural transformations: in many situations, it will be important to take direct limits of normal functors: this will be no problem as soon as the morphisms are cartesian.

2.19. Remark. Now, it is expected that we prove a result of the form: given sets A and B, one can find C (in fact $C = \mathrm{Int}(A) \cdot B$) such that $\mathrm{SET}^A \to \mathrm{SET}^B$ is equivalent to SET^C. Unfortunately, this is not the case: consider for instance the functor $F(x) = x^2$, $F(f) = f^2$ from SET to SET. There are exactly two cartesian natural transformations from F to itself, namely

$$T(x)((a, b)) = (a, b), \qquad U(x)((a, b)) = (b, a).$$

Now in any category SET^C, the number of endomorphisms of $\sum_c u_c \cdot c$ is equal to the product of the cardinals $\mathrm{card}(u_c)^{\mathrm{card}(u_c)}$, and this cannot be equal to 2.

Let us see what is wrong: cartesian natural transformations send normal forms to normal forms, and are therefore determined by their action on normal forms. Normal functors can be represented by means of power series using the normal form theorem (2.11) and here we don't use all normal forms but just equivalence classes of normal forms. Then this way of representing normal functors cannot take into account the natural transformations which just consist of changing a normal form into an equivalent one, which is precisely the case for U above.

However, this drawback, although irritating, is not as terrible as it looks: the functor App that we shall now define has enough good properties.

2.20. Definition. Let A, B be sets; the functor $\mathrm{App}^{A,B}$ is defined as a functor from $\mathrm{SET}^{\mathrm{Int}(A) \cdot B} \times \mathrm{SET}^A$ to SET^B, as follows:

$$\mathrm{App}\Big(\sum_{d,b} u_{d,b} \cdot (d, b), v\Big) = \sum_b \Big(\sum_d u_{d,b} \cdot \mathrm{SET}^A(d, v)\Big) \cdot b,$$

$$\mathrm{App}\Big(\sum_{d,b} f_{d,b} \cdot (d, b), g\Big) = \sum_b \Big(\sum_d f_{d,b} \cdot \mathrm{SET}^A(d, g)\Big) \cdot b.$$

2.21. Theorem. $\mathrm{App}^{A,B}$ *is a normal functor.*

Proof. We have not defined normal functors in two variables, although one can expect very obvious analogies with the extant theory. In any case, in view of the obvious isomorphism between $\mathrm{SET}^{\mathrm{Int}(A)\cdot B} \times \mathrm{SET}^{A}$ and $\mathrm{SET}^{\mathrm{Int}(A)\cdot B+A}$ it is clear that this makes very good sense.

It is plain that our functor is analytical. If we try to compute the coefficients of $\mathrm{App}^{A,B}$, we shall see that all coefficients are either 0 or 1. More precisely App is determined by a list of coefficients $C_{e,b}$ (as in 2.5) where $(e, b) \in \mathrm{Int}(\mathrm{Int}(A) \cdot B + A)$. It is easy to see that the only nonzero coefficients are the $C_{e,b}$'s such that e is of the form $(d, b) + d$ for some $d \in \mathrm{Int}(A)$, in which case $C_{e,b} = 1$. In fact, in the first variable, our functor is more than analytic: it is linear! □

2.22. Proposition. *Let F be a normal functor from* SET^A *to* SET^B. *Then it is possible to define an object* $u \in \mathrm{SET}^{\mathrm{Int}(A)\cdot B}$ *such that, up to isomorphism, F is the partial functor* $\mathrm{App}^{A,B}(u, \cdot) \cdot u$ *is unique up to isomorphism.*

Proof. Let G be an analytic functor isomorphic to F, and let $(K_{d,b})$ be the list of coefficients of G. Then if $u = \sum_{d,b} K_{d,b} \cdot (d, b)$, it is plain that F is isomorphic with $\mathrm{App}(u, \cdot)$. □

2.23. Remark. Let's go back to Remark 2.19: if T is a cartesian natural transformation from F to G, and if isomorphisms between F and $\mathrm{App}(u, \cdot)$, and between G and $\mathrm{App}(v, \cdot)$ have been chosen, then it is not true that we can always represent T as $\mathrm{App}(f, \cdot)$ for an appropriate $f \in \mathrm{SET}^{\mathrm{Int}(A)\cdot B}(u, v)$.

2.24. Let us give some examples of normal functors and their associated power series expansions:

(i) The functor *sum* $F(x, y) = x + y$ etc. from SET^2 to SET can be written as

$$1 \cdot \mathrm{SET}^2(e, x \cdot e + y \cdot f) + 1 \cdot \mathrm{SET}^2(f, x \cdot e + y \cdot f)$$

(e and f denoting the 'base' of SET^2).

(ii) The functor *product* $F(x, y) = x \cdot y$ etc. from SET^2 to SET can be written as

$$1 \cdot \mathrm{SET}^2(e + f, x \cdot e + y \cdot f).$$

(iii) The functor *diagonal* from SET to SET^2, $F(x) = x \cdot e + x \cdot f$ etc. can be written as $\mathrm{SET}(1, x) \cdot (e + f)$.

(iv) There are extremely many other examples of normal functors; they simply follow from the possibility of modelizing λ-calculus by means of the model A_∞ of Section 3.

3. Normal functors and λ-calculus

3.1. Proposition. *There exists a non-empty set A_∞ together with a bijection q from A_∞ onto* $\mathrm{Int}(A_\infty) \cdot A_\infty$.

Proof. Using obvious cardinality considerations, the proposition holds as soon as A_∞ is infinite. So let us ask for a little more, namely that $\mathrm{Int}(A_\infty)\cdot A_\infty$ is 'equal' to A_∞, i.e., the isomorphism q is natural: we can define A_∞ by the following inductive definition:

(i) $*\in A_\infty$.

(ii) If X is a finite subset of A_∞, f a function from X to $\mathbb{N}$-$\{0\}$, a an element of A_∞, then $(f, a)\in A_\infty$. When $a=*$, this rule is subject to the restriction that $X\neq\emptyset$.

(iii) The only elements of A_∞ are those generated by (i)–(ii).

We define the function q from A_∞ to $\mathrm{Int}(A_\infty)\cdot A_\infty$ as follows: $q((f, a))=(d, a)$ where $d=\sum_a n_a\cdot a$ with $n_a=f(a)$ if $a\in\mathrm{dom}(f)$, $n_a=0$ otherwise; $q(*)=(0, *)$ with $0=\sum_a 0\cdot a$.

It is plain that q is a bijection. □

3.2. Remark. $\mathrm{Int}(\cdot)$ can be viewed as a functor from SET to SET: when $f\in\mathrm{SET}(A, B)$

$$\mathrm{Int}(f)\Big(\sum_a n_a\cdot a\Big)=\sum n_a\cdot f(a).$$

So $\mathrm{Int}(\cdot)\cdot(\cdot)$ is a functor from SET to SET. This functor preserves direct limits, so it will not be difficult to find a lot of fixed points for it! The set A_∞ constructed in 3.1 is precisely one of them.

The functor $\mathrm{Int}(\cdot)$ also preserves pull-backs. But unfortunately it does not preserve kernels: for instance, if f is any permutation of the set $2=\{0, 1\}$, then $\mathrm{Int}(f)(\mathbf{0}+\mathbf{1})=\mathbf{0}+\mathbf{1}$ and this surely prevents the form $|\mathbf{0}+\mathbf{1}; 2; \mathrm{id}_2; 2|$ from being normal. So the basic operations on types such as the arrow will not be represented by normal functors, unless we change our pattern, and this will render the interpretation of systems involving variable types impossible in this present framework.

3.3. The model $\mathbb{A}_\infty$. Let t be a term of λ-calculus, and let $x_0, \ldots, x_{n-1}$ be a list of distinct variables including all free variables of t, which we shall also denote $t[x_0, \ldots, x_{n-1}]$. We shall construct a functor t^* from $\mathrm{SET}^{A_\infty}\times\cdots\times\mathrm{SET}^{A_\infty}$ (n times) to SET^{A_∞}. (In fact t^* will only be defined up to isomorphism.)

3.3.1. *Case of a variable.* Assume that $t[x_0, \ldots, x_{n-1}]=x_i$. Then we define

$$t^*[u_0, \ldots, u_{n-1}]=u_i, \qquad t^*[f_0, \ldots, f_{n-1}]=f_i.$$

3.3.2. *Case of an application.* Assume that $t[x_0, \ldots, x_{n-1}]=u[x_0, \ldots, x_{n-1}]$ $(v[x_0, \ldots, x_{n-1}])$. If the functors u^* and v^* have already been constructed, let q be the bijection from A_∞ onto $\mathrm{Int}(A_\infty)\cdot A_\infty$ constructed in Proposition 3.1; then q induces an isomorphism SET^q between SET^{A_∞} and $\mathrm{SET}^{\mathrm{Int}(A_\infty)\cdot A_\infty}$, hence the functor $\mathrm{SET}^q\circ u^*$ is a functor from $\mathrm{SET}^{A_\infty}\times\cdots\times\mathrm{SET}^{A_\infty}$ to $\mathrm{SET}^{\mathrm{Int}(A_\infty)\cdot A_\infty}$.

Then we use the functor $\mathrm{App}^{A_\infty,A_\infty}$ of 2.12 as follows:

$$t^*[u_0, \ldots, u_{n-1}] = \mathrm{App}(\mathrm{SET}^q(u^*[u_0, \ldots, u_{n-1}]), v^*[u_0, \ldots, u_{n-1}]),$$
$$t^*[f_0, \ldots, f_{n-1}] = \mathrm{App}(\mathrm{SET}^q(u^*[f_0, \ldots, f_{n-1}]), v^*[f_0, \ldots, f_{n-1}]),$$

and this defines a functor t^* from $\mathrm{SET}^{A_\infty} \times \cdots \times \mathrm{SET}^{A_\infty}$ to SET^{A_∞}.

3.3.3. *Case of the λ-abstractor.* Assume that $t[x_0, \ldots, x_{n-1}] = \lambda y \cdot u[x_0, \ldots, x_{n-1}, y]$. We assume that the functor u^* from $\mathrm{SET}^{A_\infty} \times \cdots \times \mathrm{SET}^{A_\infty}$ ($n+1$ times) to SET^{A_∞} has been constructed, and furthermore (this claim will be justified later on) that u^* is a normal functor. (This means that u^* enjoys the $n+1$-variable analog of normality which turns out to be equivalent to the normality of u^*, when viewed as a functor from $\mathrm{SET}^{A_\infty \cdot (n+1)}$ to SET^{A_∞}.) Then u^* admits a power series expansion:

$$u^*_a[u_0, \ldots, u_{n-1}, v]$$
$$= \sum C_{d_0 \cdots d_{n-1} e, b} \cdot \mathrm{SET}^{A_\infty}(d_0, u_0) \cdots \mathrm{SET}^{A_\infty}(d_{n-1}, u_{n-1}) \cdot \mathrm{SET}^{A_\infty}(e, v),$$

the sum being taken over all $n+1$-tuples $(d_0, \ldots, d_{n-1}, e)$ of elements of $\mathrm{Int}(A_\infty)$, with a similar expansion for morphisms. Then define the functors $t^+_{a,e}$ by:

$$t^+_{a,e} u_0, \ldots, u_{n-1} = \sum C_{d_0 \cdots d_{n-1} e} \cdot \mathrm{SET}^{A_\infty}(d_0, u_0) \cdots \mathrm{SET}^{A_\infty}(d_{n-1}, u_{n-1}),$$

the sum being taken over all n-tuples $(d_0, \ldots, d_{n-1})$ of elements of $\mathrm{Int}(A_\infty)$. Now define the functor t^+ from $\mathrm{SET}^{A_\infty} \times \cdots \times \mathrm{SET}^{A_\infty}$ (n times) to $\mathrm{SET}^{\mathrm{Int}(A_\infty) \cdot A_\infty}$ by means of its components $t^+_{a,e}$ (for $e \in \mathrm{Int}(A_\infty)$, $a \in A_\infty$). Now let r be the bijection inverse to q; we define t^* to be $\mathrm{SET}^r \circ t^+$.

3.4. Theorem. *All the functors constructed in* 3.3 *are normal.* (By the way this renders the definition 3.3 sound, since normality of the functors t^* is used in the case of λ-abstraction.)

Proof. By induction on t, we prove that t^* is normal.

(i) If t is a variable, then t^* is a projection functor, and such functors are obviously normal.

(ii) If t is $u(v)$, then t^* is obtained from u^*, v^* (normal by induction hypothesis) by means of composition with SET^q (normal because an isomorphism must be normal: it preserves everything!) and $\mathrm{App}^{A_\infty,A_\infty}$ which is normal by Theorem 2.13; then t^* must be normal.

(iii) If t is $\lambda y \cdot u$, then t^* is obtained by composition of SET^r (normal because it is an isomorphism) and t^+; t^+ is normal because we have given its explicit power series expansion.

3.5. Substitution Lemma. *Let $t[x_0, \ldots, x_{n-1}]$, $w_0[y_0, \ldots, y_{m-1}], \ldots, w_{n-1}[y_0, \ldots, y_{m-1}]$ be terms of the λ-calculus, and let $u[y_0, \ldots, y_{m-1}] = t[w_0, \ldots, w_{n-1}]$ be the* *term obtained by substituting the w_i's for the x_i's in t. Then u^* is (isomorphic to) the result of composing t^* with the w_i^*'s*:

$$u^*[u_0, \ldots, u_{m-1}] = t^*[w_0^*[u_0, \ldots, u_{m-1}], \ldots, w_{n-1}^*[u_0, \ldots, u_{m-1}]] \quad etc.$$

Proof. By induction on the term t:

(i) t is x_i; then u is w_i etc.

(ii) t is $t'(t'')$; then u is of the form $u'(u'')$, and we can assume that the induction hypothesis holds for u' and u'', e.g.,

$$u'^*[u_0, \ldots, u_{m-1}] = t'^*[w_0^*[u_0, \ldots, u_{m-1}], \ldots, w_{n-1}^*[u_0, \ldots, u_{m-1}]] \quad \text{etc.}$$

Such equations will persist if we apply SET^q to both sides, and similarly if we apply $\mathrm{App}^{A_\infty, A_\infty}$ to both sides.

(iii) t is $\lambda z t'$; in order to simplify the expressions, let $n = m = 1$; the induction hypothesis yields, with $u'[y_0, y_1] = t'[w[y_0], y_1]$: $u'^*[u_0, u_1] = t'^*[w^*[u_0], u_1]$. Now,

$$t_a'^*[v_0, v_1] = \sum C_{d_0 d_1, a} \cdot \mathrm{SET}^{A_\infty}(d_0, v_0) \cdot \mathrm{SET}^{A_\infty}(d_1, v_1) \quad \text{and}$$

$$u_a'^*[u_0, u_1] = \sum C_{d_0 d_1, a} \cdot \mathrm{SET}^{A_\infty}(d_0, w^+[u_0]) \cdot \mathrm{SET}^{A_\infty}(d_1, v_1).$$

Now $t_{a,e}'^+(v_0)$ is the coefficient of $\mathrm{SET}^A(e, v_1)$ in the expansion of $t_a'^*[v_0, v_1]$ and similarly, $u_{a,e}'^+(u_0)$ is the coefficient of $\mathrm{SET}^A(e, u_1)$ in the expansion of $u_a'^*[u_0, u_1]$, and this forces the equality (in fact: isomorphism) $u_{a,e}'^+(u_0) = t_{a,e}'^+(w^*[u_0])$ (and the same for morphisms). Then, by applying SET^r to both sides, one gets $u^*[u_0] = t^*[u_0]$ etc. □

3.6. Theorem. *Assume that the terms t and u are such that $t \dashv u$. Then t^* and u^* are isomorphic functors.*

Proof. In the definition of $t \dashv u$, we have the iteration of several atomic possibilities of reduction. Most of them obviously preserve the associated functors, typically so-called 'α-conversion', i.e., change of bound variables, and also the rules expressing the compatibility of the operations $\cdot(\cdot)$ and $\lambda y.$ with conversion, not to speak of transitivity of conversion. The only real problem is with so called 'β-conversion', that is $\lambda y \,.\, t[y](u) \dashv t[u]$. So we must show that $(\lambda y t[y](u))^* = t^* \circ_y u^*$. We shall do it in the special case where the λ-term has at most one free variable x_0: assume that

$$t_a^*[u_0, u_1] = \sum C_{d_0 d_1, a} \cdot \mathrm{SET}^{A_\infty}(d_0, u_0) \cdot \mathrm{SET}^{A_\infty}(d_1, u_1) \quad \text{etc.},$$

then we must show that $(\lambda y \,.\, t[y](u))^*[u_0]$ is naturally isomorphic to

$$\sum_a \Big(\sum_{d,e} C_{de,a} \cdot \mathrm{SET}^{A_\infty}(d, u_0) \cdot \mathrm{SET}^{A_\infty}(e, u^*[u_0]) \Big) \cdot a \quad (= F(u_0) \text{ etc.})$$

Then $(\lambda y t[y])^{+}_{a,e}[u_0] = \sum_d C_{de,a} \cdot \mathrm{SET}^A(d, u_0)$ etc. and $(\lambda y \,.\, t[y])^* = \mathrm{SET}^q \circ (\lambda y t)^+$. In order to form $(\lambda y t(u)^*$ one first makes the composition $\mathrm{SET}^q \circ (\lambda y t)^*$, which yields $(\lambda y t)^+$; then one forms $\mathrm{App}^{A_\infty, A_\infty}((\lambda y t)^+[u_0], u^*[u_0])$ and we obtain

$$\sum_a \Big(\sum_e (\lambda y t)^{+}_{a,e} \cdot \mathrm{SET}^{A_\infty}(e, u^*[u_0]) \Big) \cdot a$$

and this expression is isomorphic to $F(u_0)$ in a natural way. □

3.7. Remarks. (i) So called 'η-conversion' is also valid for our interpretation.

(ii) The model constructed is non-trivial: for instance $(\lambda x \,.\, \lambda y x)^*$ and $(\lambda x \,.\, \lambda y y)^*$ are not isomorphic.

(iii) t^* is defined up to isomorphism: so if we want t^* to be well-defined, the coefficients of t^* will be cardinals (very often integers). So a typical question is to determine the behaviour of the coefficients of t^* w.r.t. the set A and the bijection q.

(iv) Let us explicitly write the interpretation of λ and AP, taking into account the specifity of our solution (A_∞, q) found in 3.1; from the identification between $\mathrm{Int}(A_\infty) \cdot A_\infty$, one can deduce a binary function $(\cdot,\cdot)$ from $\mathrm{Int}(A_\infty)$ and A_∞ to A_∞:

– It is clear that functions from finite subsets of A_∞ into $\mathbb{N}$-$\{0\}$ can be identified with elements of $\mathrm{Int}(A_\infty)$. So when $d \in \mathrm{Int}(A_\infty)$, $c \in A_\infty$, (d, c) is a well defined point of A_∞, except when $d = 0$ and $c = *$ in which case we set $(0, *) = *$.

– The functor App from $\mathrm{SET}^{A_\infty} \times \mathrm{SET}^{A_\infty}$ to SET^{A_∞} is defined by

$$\mathrm{App}\Big(\sum_a u_a \cdot a, v\Big) = \sum_{d \in \mathrm{Int}(A_\infty)} \mathrm{SET}^A(d, v) \cdot \Big(\sum_{a \in A_\infty} u_{(d,a)} \cdot a\Big)$$

– Conversely, any normal functor F from SET^{A_∞} to SET^{A_∞} can be written

$$F(v) = \sum_{d \in \mathrm{Int}(A_\infty)} \mathrm{SET}^{A_\infty}(d, v) \cdot \Big(\sum_{a \in A_\infty} u_{d,a} \cdot a\Big) \quad \text{etc.}$$

so the functor F can be encoded by the object $\sum_{d,a} u_{d,a} \cdot (d, a)$.

3.8. Definition. Let (A, q) be the pair of a non-empty set A together with a bijection q from A onto $\mathrm{Int}(A) \cdot A$; to some elements of A, we attribute a level:

If $q(a) = (n_1 \cdot a_1 + \cdots + n_k \cdot a_k, b)$ and if $a_1, \ldots, a_k$ have levels $l(a_1), \ldots, l(a_k)$, then $l(a)$ is defined and $l(a) = \sup(l(a_i) + 1)$. For instance the points of level 0 are the points $q^{-1}((0, b))$.

(A, q) is said to be *regular* if all the elements of A have a level. The explicit pair (A_∞, q) of 3.1 is regular.

3.9. Definition. Let (A, q) be a pair as in 3.8; for any integer n, we define a normal functor p_n from SET^A to itself by:

$$p_n\Big(\sum_{a \in A} u_a \cdot a\Big) = \sum_{a \in A_n} u_a \cdot a \quad \text{etc.}$$

where A_n is the subset of A formed of those points of level $< n$.

3.10. Proposition. (i) $p_n \circ p_m = p_{\inf(n,m)}$.
(ii) $p_0(x) = 0$ *etc.*
(iii) $p_{n+1}(x) = \mathrm{SET}^r(\mathrm{SET}^q(x) \circ p_n)$ *etc.*

(In $\mathrm{SET}^q(x) \circ p_n$ we compose an object of $\mathrm{SET}^{\mathrm{Int}(A) \cdot A}$ with a normal functor from SET^A to itself: this means that we must replace $\mathrm{SET}^q(x)$ by the functor it encodes, make the composition, then find a point in $\mathrm{Set}^{\mathrm{Int}(A) \cdot A}$ encoding the result.)

Proof. (i) $A_0 \subset A_1 \cdots \subset A_n$ and p_n is just the projection on SET^{A_n}.
(ii) is immediate since $A_0 = \emptyset$.
(iii) If we define a normal functor p by $p(x) = \mathrm{SET}^r(\mathrm{SET}^q(x) \circ p_n)$, then

$$p(p(x)) = \mathrm{SET}^r(\mathrm{SET}^q(x) \circ p_n \circ p_n)) = p(x)$$

since p_n is a projector. Then to prove that p and p_{n+1} coincide will amount to prove that $p(a) = p_{n+1}(a)$ for all $a \in A$:

(1) If $l(a) \leqslant n$, this means that $q(a) = (n_1 \cdot a_1 + \cdots + n_k \cdot a_k, c)$ with $l(a_i) < n$; then $p_{n+1}(a) = a$; but $\mathrm{SET}^q(a) = q(a)$ encodes the functor $F(\sum_{b \in A} u_b \cdot b) = u_{a_1}^{n_1} \cdots u_{a_k}^{n_k} \cdot c$ etc. and since $p_n(a_i) = a_i$ it follows that $F \circ p_n = F$; then $\mathrm{SET}^r(\mathrm{SET}^q(a) \circ p_n) = a$.

(2) Otherwise, $q(a) = (n_1 \cdot a_1 + \cdots + n_k \cdot a_k, c)$, and one of the a_i's has no level or a level $l(a_i) \geqslant n$; then $p_{n+1}(a) = 0$; $\mathrm{SET}^q(a)$ defines a functor F as above, but now $p_n(a_i) = 0$ for some i, which means that $p(u_{a_1})^{n_1} \cdots p(u_{a_k})^{n}k \cdot c$ will be zero: then $p(a) = 0$. □

3.11. Proposition. *Let* p_ω *be the projector mapping* SET^A *onto its subcategory* SET^{A_ω} *where* $A_\omega = \{a \in A; l(a)$ *is defined*$\}$. *When* $i \leqslant j \leqslant \omega$ *we define* t_{ij} *a natural transformation from* p_i *to* p_j *by*:

$$t_{ij}\left(\sum_{a \in A} u_a \cdot a\right) = \sum_{a \in A} u_a \cdot f_a \cdot a$$

where f_a *is the identity of* $1a$ *when* $a \in A_i$, f_a *is the only function from* 0 *to* 1 *when* $a \in A_j - A_i$, f_a *is the only member of* SET(0, 0) *when* $a \notin A_j$. *Then*

$$(p_\omega, t_{n\omega}) = \lim_{n \in \mathbb{N}} (p_n, t_{nm}).$$

In particular, if (A, q) *is regular, the identity of* SET^A *will be the direct limit of the* p_n*'s.*

Proof. More or less immediate: this says that A_ω is the union of the A_n's.

3.12. Theorem. *Let* (A, q) *be a regular pair. Then consider the two fixed point operators defined on* SET^A:

(i) *If u is a vector of* Set^A, *then* SET$^q(u)$ *encodes a normal functor F from* SET^A *to* SET^A; *then we can consider the vectors* $0, F(0), F(F(0)), \ldots, F^n(0), \ldots$ *When* $n \leq m$, *it is possible to define* $f_{nm} \in \mathrm{SET}^A(F^n(0), F^m(0))$ *by*: $f_{nm} = F^n(f_{0m-n})$, *the morphisms* f_{0p} *being clearly defined since their source is* 0. *Then define* $(F^\omega(0), f_{n\omega}) = \varinjlim(F^n(0), f_{nm})$. *We put* $Y_{\mathrm{ext}}(u) = F^\omega(0)$. *The definition extends to the case where* $g \in \mathrm{SET}^A(u, u')$; *in fact* Y_{ext} *is the direct limit of the functors*

$$Y^n(u) = (\mathrm{SET}^q(u) \circ \cdots \circ \mathrm{SET}^q(u))(0) \quad \textit{etc.}$$

Y_{ext} is normal, because it is easily checked that the natural transformations defined by means of the f_{nm}'s are cartesian.

(ii) $Y_{\mathrm{int}}(u) = (\lambda z\,.\,u(z(z)))(\lambda z\,.\,u(z(z)))$ *etc.*

These two fixed point operators coincide.

Proof. We shall for instance show that $Y_{\mathrm{int}}(u) = Y_{\mathrm{ext}}(u)$ when $u \in \mathrm{Set}^A$; then we write $Y_{\mathrm{int}}(u) = H(H)$. By 3.11, $(H, t_{n\omega}(H)) = \varinjlim(p_n(H), t_{nm}(H))$ and since application of λ-calculus is represented by a normal functor,

$$(H(H), t_{n\omega}(H)(t_{n\omega}(H))) = \varinjlim(p_n(H)(p_n(H)), t_{nm}(H)(t_{nm}(H))).$$

(i) $p_0(H) = 0$, so $p_0(H)(p_0(H)) = 0 = F^0(0)$.

(ii) Assume that $p_n(H)(p_n(H)) = F^n(0)$. Then

$$p_{n+1}(H)(p_{n+1}(H)) = (H \circ p_n)(p_{n+1}(H)) = H(p_n(p_{n+1}(H)) = H(p_n(H));$$

but the reduction rules are valid so $H(p_n(H)) = u(p_n(H)(p_n(H))) = F^{n+1}(0)$.

(iii) For similar reasons, $t_{nm}(H)(t_{nm}(H)) = f_{nm}$.

Then the direct systems $(F^n(0), f_{nm})$ and $(p_n(H)(p_n(H)), t_{nm}(H)(t_{nm}(H)))$ coincide, so their direct limits are the same. □

3.13. Remark. The two fixed points operations Y_{int} and Y_{ext} are defined for arbitrary pairs (A, q); both satisfy the fixed point equation $u(Y(u)) = Y(u)$ etc.

But in general they do not coincide. What remains is that Y_{ext} is a minimum solution to the fixed point equation: concretely this means that there is a cartesian natural transformation from Y_{ext} to Y_{int}, T, such that $u(T(u)) = T(u)$. Y is uniquely determined by this condition.

4. Normal functors and recursion theory

Consider the set $B = \{N_0, \ldots, N_n, \ldots\}$ where the N_i's are just distinct symbols for the integers $0, 1, 2, \ldots$ What is the possible meaning of the category SET^B and of the related normal functors in terms of recursion theory?

4.1. Meaning of the basis vectors. Certainly the vector N_i represents the integer

i. In particular when a normal functor F is such that $F(N_i) = N_{f(i)}$ for some function f, one can say that F represents the function f. Observe that several normal functors may represent the same function.

The zero vector 0 represents the absence of information, i.e., what corresponds to 'undeterminated' in usual partial recursion. A normal functor can represent a partial function f, provided we require that $F(N_i) = 0$ when $f(i)$ is undeterminated.

4.2. Meaning of arbitrary objects. What about now a linear combination such as $N_5 + N_{17}$? This indicates the *superposition* of the situations corresponding to N_5 and N_{17}. This can happen in the practice of computation: for instance we have an input N somewhere and

(i) Either our indications are contradictory (one line says $N = 5$, another says $N = 17$),

(ii) Or we may think of a probabilistic algorithm: the input N is determined by a random process, independent of the program, and the two possible values for N are 5 and 17. Here we are not interested in the relative weights of the inputs 5 and 17.

(iii) We can also think of a parallel algorithm: one processor tries to find the answer using the value $N = 5$, the other tries to find the answer using the value $N = 17$.

Now F is a normal functor from SET^B to SET^B; what is the meaning of the value $F(N_5 + N_{17})$? Say for instance that $F(N_5 + N_{17}) = N_4 + N_{20} + N_{32}$:

(i) This can indicate that our results are contradictory, just as our inputs.

(ii) If we are working with a probabilistic algorithm, this means that, according to the random choices made, the values obtained are 4, 20 and 32. Of course more than two values can occur because we may ask several times for the value of N, and we are not forced to give the same answer.

(iii) In terms of parallel computation, $N_4 + N_{20} + N_{32}$ just counts all the outputs that come from the independent lines of program that work parallelly.

But what about an input (or an output) of the form $2 \cdot N_5 + 6 \cdot N_{17}$? In general this indicates as before a superposition of situations, but we take care of the *order of multiplicity*. The situation where this is the most easy to understand is the case of a parallel program: the output $2 \cdot N_5 + 6 \cdot N_{17}$ means that two independent lines of computation led to the value 5 whereas six other lines of computation led to the value 17.

If we understand finite objects, there is not problem in understanding infinite ones.

4.13. The principle of positive information. This is just the remark that if we think of normal functors as effective operations, then functions and arguments appear as (recursive) direct limits of finite functions and arguments. Then the value $F(x)$ also appears as a recursive direct limit. It is of course impossible to predict anything negative like "coefficient u_n is zero" because at no finite stage we can be sure that u_n is zero forever. But positive information of the kind "u_n is nonzero" or "u_n has at least p elements" can be effectively checked when they are true. This kind of fact is familiar from the theory of partial recursive functions and recursive enumerability.

For instance, if we want to define a function using the line "if N is nonzero, then ..." it is impossible to use the coefficient of N_0 for that purpose; but one can use the other coefficients. We shall see later that it is even more appropriate to introduce a specific vector U_0 which precisely says that the input is nonzero.

4.4. Example. Consider the well-known parallel definition:

$$f(0, x) = 0, \qquad f(x, 0) = 0, \qquad f(n+1, m+1) = 1.$$

We propose to represent this function by the following binary normal functor F:

$$\begin{aligned} &F(n_0N_0 + \cdots + n_pN_p + \cdots, m_0N_0 + \cdots + m_pN_p + \cdots) \\ &\quad = (n_0 + m_0)N_0 + (n_1 + \cdots + n_p + \cdots)(m_1 + \cdots + m_p + \cdots)N_1 \quad \text{etc.} \end{aligned}$$

$$F(N_0, 0) = F(0, N_0) = N_0, \quad \text{but} \quad F(N_0, N_0) = 2N_0,$$

in other terms, the definition takes into account the fact that there are two ways of computing $f(0, 0)$: the order of multiplicity is 2.

The coefficient of N_1 in the definition of F is simply the following: assume that we want to evaluate $F(3 \cdot N_4 + 7 \cdot N_9, N_1 + 2 \cdot N_7)$; then the third defining equation may be used between 4 and 1 three times, between 9 and 1 seven times, between 3 and 7 six times, between 9 and 7 fourteen times: so the answer is 1 (i.e., N_1), with the multiplicity $3 + 7 + 6 + 14 = 30 : 30N_1$.

4.5. Definition. We define a set K by $K = \{N_0, U_0, N_1, U_1, N_2, U_2, \ldots\}$ where the N_i's and U_i's are distinct symbols.

Although it is prefectly possible to develop recursion on the basis of the set B so far considered, this leads to somewhat inelegant facts. For instance it is not possible to obtain any functor F from SET^B to itself such that $F(N_0) = N_0$, $F(\mathrm{Sh}(x)) = N_1$ where Sh is the shift functor (because $\mathrm{Sh}(0) = 0$ but N_1 is not a subobject of N_0). So, besides the N_i's we shall add additional vectors, the U_i's: the meaning of U_i alone is "the datum is strictly greater than i". In particular U_0 will be a positive way of saying that something is nonzero.

4.6. Definition. We define the following normal functors from SET^K to itself:

(i) $$\begin{aligned} &T^*(n_0 \cdot N_0 + u_0 \cdot U_0 + n_1 \cdot N_1 + u_1 \cdot U + \cdots) \\ &\quad = n_0 \cdot N_1 + u_0 \cdot U_1 + n_1 \cdot N_2 + u_1 \cdot U_2 + \cdots \quad \text{etc.} \end{aligned}$$

(ii) $$S^*(x) = U_0 + T^*(x) \quad \text{etc.}$$

The objects n^* of SET^K are defined by:

$$0^* = N_0, \qquad (n+1)^* = S^*(n^*),$$

hence $n^* = U_0 + U_1 + \cdots + U_{n-1} + N_n$.

4.7. Examples. (i) The constant functions, the projection functions, the successor function can be represented by normal functors. For instance the successor function can be represented by S^*. 'Represented' means that $(\mathrm{succ}(n))^* = S(n^*)$ for all n.

(ii) The class of functions that can be represented by (effective) normal functors is closed under composition.

(iii) Consider the defining equation (for simplicity, only one auxiliary variable x has been used):

$$f(x, 0) \simeq g(x), \qquad f(x, \mathrm{succ}(y)) \simeq h(x, f(x, y), y)$$

and assume that g and h are already represented by G and H. Then observe that any object y of SET^K can be uniquely written as $a \cdot N_0 + b \cdot U_0 + T^*y'$ for some sets a and b and some $y' \in \mathrm{Set}^K$. Then we define F by means of the equation

$$F(x, a \cdot N_0 + b \cdot U_0 + T^*y') = a \cdot G(x) + b \cdot H(x, F(x, y'), y') \quad \text{etc.}$$

This equation enables us to compute $F(x, y)$ when y has finitely many nonzero coefficients, and in general via direct limits.

This functor represents F: when $f(n, m)$ is defined then, $F(n^*, m^*) = f(n, m)^*$.

(iv) Consider the defining equation (for simplicity, of a unary function):

$$f(x) \simeq \mu y\,.\,(g(x, y) \simeq 0)$$

and assume that g has been represented by the normal functor G. Write

$$G(x, p^*) = n_0^p(x) \cdot N_0 + u_0^p(x) \cdot U_0 + n_1^p(x) \cdot N_1 + u_1^p(x) \cdot U_1 + \cdots \quad \text{etc.}$$

Then define the normal functor F by:

$$\begin{aligned} F(x) = {} & n_0^0(x) \cdot N_0 + u_0^0(x) \cdot U_0 + u_0^0(x) \cdot n_0^1(x) \cdot N_1 \\ & + u_0^0(x) \cdot u_0^1(x) \cdot U_1 + u_0^0(x) \cdot u_0^1(x) \cdot n_0^2(x) \cdot N_2 \\ & + u_0^0(x) \cdot u_0^1(x) \cdot u_0^2(x) \cdot U_2 + \cdots \quad \text{etc.} \end{aligned}$$

In other terms

$$\begin{aligned} F(x) = {} & n_0^0(x) \cdot N_0 + u_0^0(x) \cdot (U_0 + n_0^1(x) \cdot N_1 + u_0^1(x) \cdot (U_1 + n_0^2(x) \cdot N_2 \\ & + u_0^2(x) \cdot (U_2 + \cdots \end{aligned}$$

It is immediate that F represents f. Maybe it could be of some interest to explain how we came to such coefficients:

– The coefficient $n_0^0(x)$ of N_0 gives us the number of lines of computation that lead to $g(x, 0) = 0$, the number of times we know that $g(x, 0) = 0$.

– The coefficient $u_0^0(x)$ of U_0 tells us how many times we know that $g(x, 0) \neq 0$: each time we know this, we deduce that $f(x) \neq 0$.

– In order to conclude that $f(x) = 1$, we must know that $g(x, 0) \neq 0$ (we know it $u_0^0(x)$ times) and $g(x, 1) = 0$ (we know it $n_0^1(x)$ times): this explains the coefficient of $N_1 : u_0^0(x) \cdot n_0^1(x)$.

4.8. Remarks. (i) The use of the vectors U_i comes from the fact that we do not want S^* to leave 0 unmoved; this is quite natural, because in a computation, when we know that the result is the successor of something, even if we are not able to compute this something, we have got some information. One could consider other basis vectors, corresponding to other kinds of information on integers.

(ii) S^* has exactly one fixed point, namely $U_0 + U_1 + U_2 + \cdots$ This object is one of the most typical possible outputs: for instance if we want to compute X_0, and at some stage we discover that $X_0 = \mathrm{succ}(X_1)$, then trying to compute X_1, we obtain $X_1 = \mathrm{succ}(X_2)$ etc., then the result should be represented by $U_0 + U_1 + U_2 + \cdots$

But this infinite object (let us call it U_∞) is even more interesting as an input: in Example 4.7(ii) if we try to evaluate $F(x, U_\infty)$ we get: $F(x, U_\infty) = H(x, F(x, U_\infty), U_\infty)$. In particular, when H does not depend on its last argument, this means that $F(x, U_\infty) = H(x, F(x, U_\infty))$ etc. and so $F(x, U_\infty)$ is just the smallest fixed point of the functor $H(x, \cdot)$.

(iii) We have represented recursive functions by recursive normal functors. It is plain that the representation takes into account the way the function has been defined. For instance the binary functor representing the function sum is not symmetric. It seems reasonable that the semantic of functions keeps some trace of the original algorithm, provided this trace is not some kind of fuzzy 'intensionality', but as we do it here, some simple and manageable structural information.

(iv) It is not very clear which kind of recursive functions should be interpreted by means of normal functors: clearly all partial recursive algorithms have a simple and natural interpretation. But one can also think of algorithms

– that allow parallel possibilities of computing the values, such as the one considered in 4.4.

– We can even imagine algorithms giving inconsistent answers, for instance

$$f(0, x) = 0, \qquad f(x, 0) = 1.$$

The normal functor $F(n_0 \cdot N_0 + \cdots, m_0 \cdot N_0 + \cdots) = n_0 \cdot N_0 + m_0 \cdot U_0 + m_0 \cdot N_1$ takes care of this algorithm without any problem. Now is there any real interest in developing the recursion theory of such algorithms? If there is any this seems to be a rather starightforward matter.

5. Normal functors and functionals of finite type

5.1. Definition. The finite types are given by:

(i) *int* is a finite type.

(ii) If σ and τ are finite types, then $\sigma \times \tau$ is a finite type.

(iii) If σ and τ are finite types, then $\sigma \to \tau$ is a finite type.

(iv) The only finite types are those given by (i)–(iii).

5.2. Definition. Let σ be a finite type (for short, let us say 'a type'); then we define a set σ^* as follows:

(i) $int^* = K$ (the set considered in 4.5),

(ii) $(\sigma \times \tau)^* = \sigma^* + \tau^*$,

(iii) $(\sigma \to \tau)^* = \mathrm{Int}(\sigma^*) \cdot \tau^*$.

5.3. Definition. If σ is a type, then the category of objects and morphisms 'of type σ' will in fact be SET^{σ^*}.

5.4. Remark. $(\sigma \times \tau)^*$ is a disjoint sum; this means that we must first replace σ^* and τ^* by disjoint isomorphic sets, typically $\{0\} \times \sigma^*$ and $\{1\} \times \tau^*$, and then take their union. In practice, we shall always implicitely assume that σ^* and τ^* are disjoint, so that $(\sigma \times \tau)^*$ will be their union. The reader will reconstitute himself the correct underlying construction.

5.5. Interpretation of Gödel's $\mathcal{T}$ (I). Here we concentrate on the part of Gödel's $\mathcal{T}$ which has nothing to do with the intended meaning of the type *int*, i.e., the terms built from the variables by means of application, λ-abstraction, pairing and projection, in a way compatible with the type structure.

Assume that t $(= t[x_1, \ldots, x_n])$ is a term of type τ, and that $x_1, \ldots, x_n$ are distinct variables of respective types $\sigma_1, \ldots, \sigma_n$ including all free variables of t; then we define a functor t^* from $\mathrm{SET}^{\sigma_1^*} \times \cdots \times \mathrm{SET}^{\sigma_n^*}$ to SET^{τ^*} as follows:

5.5.1. *Case of a variable.* t is x_i; then t^* is the projection functor $t^*[u_1, \ldots, u_n] = u_i$ etc.

5.5.2. *Case of an application.* Assume that t and u are of respective types $\theta \to \rho$ and θ; then one defines $(t(u))^*$ by

$$(t(u))^*[v_1, \ldots, v_n] = \mathrm{App}^{\theta^*, \rho^*}(t^*[v_1, \ldots, v_n], u^*[v_1, \ldots, v_n]) \quad \text{etc.}$$

5.5.3. *Case of a λ-abstraction.* If t is of type ρ, $t = t[x_1, \ldots, x_n, y]$, and y is of type θ, then we have already constructed t^* admitting the power series expansion:

$$t_a^*[u_0, \ldots, u_n, v] = \sum C_{d_1 \cdots d_n e, a} \cdot \mathrm{SET}^{\sigma_1^*}(d_1, u_1) \cdots \mathrm{SET}^{\sigma_n^*}(d_n, u_n) \cdot \mathrm{SET}^{\theta^*}(e, v)$$

and $(\lambda yt)^*$ is defined as expected by its components on couples (e, a):

$$(\lambda yt)^*_{e,a}[u_1, \ldots, u_n] = \sum C_{d_1\cdots d_n e,a} \cdot \mathrm{SET}^{\sigma_1^*}(d_1, u_1) \cdots \mathrm{SET}^{\sigma_n^*}(d_n, u_n) \quad \text{etc.}$$

5.5.4. *Case of a couple.* With the abuse of notations of 5.4, assume that t, u are of respective types θ and ρ; then the components of $(t, u)^*$ are

$$(t, u)^*_a = t^*_a \quad \text{when } a \in \theta^*,$$
$$(t, u)^*_b = u^*_b \quad \text{when } b \in \rho^*.$$

5.5.5. *Case of a projection.* Assume that t is of type $\theta \times \rho$; then

$$(\pi^1 t)^*_a = t^*_a \quad \text{for } a \in \theta^*,$$
$$(\pi^2 t)^*_b = t^*_b \quad \text{for } b \in \rho^*.$$

5.6. Theorem. *If two terms are interconvertible, then they have the same interpretation (up to isomorphism).*

Proof. This is similar to the result already obtained for λ-calculus (3.6). By the way observe that the 'η-rules' $\lambda y . a(y) \dashv a$ and $(\pi^1 a, \pi^2 a) \dashv a$ are sound for this interpretation. □

5.7. Theorem. *Assume that x is an object of* SET^A, *and let H be a normal functor from* $\mathrm{SET}^A \times \mathrm{SET}^K$ *to* SET^A. *Then it is possible to find a normal functor F from* SET^K *to* SET^A *satisfying the equation*:

$$F(a \cdot N_0 + b \cdot N_1 + T^*(v)) = a \cdot x + b \cdot H(F(v), v) \quad \textit{etc.}$$

Proof. Let $\mathscr{C}^k$ be the category $\mathrm{SET}^{\{N_0, U_0, \ldots, N_{k-1}, U_{k-1}\}}$. Then using the defining equation for F, we can obtain normal functors F^k from $\mathscr{C}^k$ to SET^A:

(i) $F^0(0) = 0$ etc.

(ii) $F^{k+1}(a \cdot N_0 + b \cdot N_1 + T^*(v)) = a \cdot x + b \cdot H(F^k(v), v)$ etc.

By trivial considerations on normal functors, it is immediate that the functors F^k are normal.

Now $\mathscr{C}^0 \subset \mathscr{C}^1 \subset \cdots \subset \mathscr{C}^k \subset \cdots \subset \mathrm{SET}^K$; the functors F^k extend one another. Then we can define a functor F^ω on the union $\mathscr{C}^\omega$ of the categories $\mathscr{C}^k$, corresponding to those objects and morphisms whose coefficients are almost all zero: F^ω is simply the 'union' of all the F^k's.

The crucial point will be to determine the relationship between normal forms w.r.t. F^k and w.r.t. F^{k+1}: Let $a \in A$ and assume that $z = (x^*; \bar{d}; \bar{f}; \bar{v})_{F^k_a}$ is the normal form of $z \in F^a_k(v)$; then $z = |z^*; d; f; v|_{F^{k+1}_a}$. We show that this form is normal too: Assume that $z = |y^*; e; g; v|_{F^{k+1}_a}$; then by a pull-back argument (recall that F^{k+1} is normal) it is possible to find c, x^*, f' and g' such that $ff' = gg'$ and $z^* = |x^*; c; f'; d|_{F^{k+1}_a}$ and $y^* = |x^*; c; g'; e|_{F^{k+1}_a}$. The existence of the morphism f'

forces c to belong to the category $\mathscr{C}^k$, so $z^* = |x^*; c; f'; d|_{F_a^k}$ and we can assume without loss of generality that $f' = i_d$, $c = d$, $x^* = z^*$. Then $f = gg'$ and $y^* = F_a^{k+1}(g')(z^*)$. It remains to prove that g' is uniquely determined by the conditions. Assume that $e = n_0 \cdot N_0 + u_0 \cdot U_0 + \cdots + u_k \cdot U_k$. Assume that $g' = f_0 \cdot N_0 + g_0 \cdot U_0 + \cdots + g_k \cdot U_k$; since f_k and g_k have the source 0, they are completely determined. Let $g'' = f_0 \cdot N_0 + \cdots + g_{k-1} \cdot U_{k-1}$, $h = i_{n_0} \cdot N_0 + \cdots + i_{u_{k-1}} \cdot U_{k-1} + f_k \cdot N_k + g_k \cdot U_k$; then $g' = hg''$ and one can write $y^* = F_a^{k+1}(h)(y^{**})$ with $y^{**} = F_a^{k+1}(g'')(z^*) = F_a^k(g'')(z^*)$. y^{**} is well determined because of the injectivity of h and consequently of $F_a^{k+1}(h)$; g'' is uniquely determined for normal form reasons, and we can conclude that $g' = hg''$ is uniquely determined.

So we have established that normal forms w.r.t. F_a^k remain normal forms w.r.t. F_a^{k+1}. This proves that the functor F^ω is normal (although we have not defined normality on categories such as $\mathscr{C}^\omega$). Then F is simply defined on SET^K by means of a direct limit extension. (For instance, it is possible to compute the 'power series expansion' of F^ω, and F is simply the analytic functor from SET^K to SET^A with the coefficients of F^ω.) □

5.8. Remark. To be rigorous, we only obtain an isomorphism between the functors corresponding to both sides of the equation in Theorem 5.7.; the equality holds when the arguments have only finitely many nonzero coefficients. In practice the equation must be taken as an equality: for instance in Gödel's $\mathscr{T}$ where all objects have weakly finite interpretations (see next section), one can assume that t^* is with integer coefficients, and then equalities such as in 5.7 are necessarily fulfilled, since isomorphism between analytic functors with integer coefficients implies equality.

5.9. Interpretation of Gödel's $\mathscr{T}$ (II). We now complete the interpretation given in 5.5.

5.9.1. *Case of* 0. 0^* is the constant N_0.

5.9.2. *Case of S.* We have defined a normal functor S^* from SET^K to itself; then this functor can be encoded by a vector (still denoted S^*) of $\mathrm{SET}^{\mathrm{Int}(K)\cdot K}$, namely

$$S^* = (0, U_1) + (N_0, N_1) + (U_0, U_1) + (N_1, N_2) + (U_1, U_2) + \cdots$$

5.9.3. *Case of a recursion.* Assume that t and u are closed terms of respective types σ and $\sigma \rightarrow (int \rightarrow \sigma)$; then the normal functor $(Rtu)^*$ is defined by means of the equation:

$$(Rtu)^*(a \cdot N_0 + b \cdot U_1 + T^*v) = a \cdot t^* + b \cdot u^+((Rtu)^*(v), v) \quad \text{etc.}$$

where u^+ is the normal functor from $\mathrm{SET}^{\sigma^*} \times \mathrm{SET}^K$ to SET^{σ^*} defined by $u^+(v, w) = \mathrm{App}(\mathrm{App}(u^*, v), w)$ etc.

5.10. Theorem. *Interconvertible terms have isomorphic interpretations.*

Proof. This is a complement to Theorem 5.6. The new case is that of a recursion; but the interpretation of the rules

$$(Rtu)(0) \neq t, \qquad (Rtu)(Sv) \neq u((Rtu)(v), v)$$

is immediate. □

5.11. Remarks. (i) Without adding the U_i's, it would be impossible to satisfy the defining equations of recursion with free variables.

(ii) Computer scientists often work with a variant of Gödel's $\mathcal{T}$, where recursion is replaced by a fixed point operation. This system is stronger than the system with recursion only, because recursion can be obtained as a fixed point. Conversely, the fixed point can be obtained by evaluating recursion at $U_0 + U_1 + U_2 + \cdots$ (4.8(ii)).

6. Finiteness properties

In complex analysis, when we evaluate an analytic function at some argument, the essential question is that of the convergence of the defining series; in our framework, there is also a notion of convergence, of a very trivial nature, namely a direct system of sets converges when its limit is finite. Then an analytic function from SET^A to SET, with integer coefficients, evaluated at an argument x, with integer coefficients, will converge just in case the infinite sum defining the value is indeed finite because almost all summands are null. Of course, depending on which arguments we want our function to converge, this will impose various conditions on the coefficients of the function.

The two extreme conditions that we can ask on the coefficients of a vector are the following:

– Finiteness, i.e., to have finitely many nonzero coefficients, all of them finite.

– Weak finiteness, i.e., to have finite coefficients.

In practice, finiteness is a too strong requirement, whereas weak finiteness is too liberal.

6.1. Definition. (i) $!\mathrm{Set}^A$ is the class of all finite objects of SET^A.

(ii) $?\mathrm{Set}^A$ is the class of all weakly finite objects of SET^A.

6.2. Theorem. (i) $!\mathrm{Set}^{\mathrm{Int}(A)\cdot B}(?\mathrm{Set}^A) = !\mathrm{Set}^B$.

(ii) $!\mathrm{Set}^A \to ?\mathrm{Set}^B \subset ?\mathrm{Set}^{\mathrm{Int}(A)\cdot B}$.

Proof. (i) The first equation means that if we apply a finite vector F of $\mathrm{Set}^{\mathrm{Int}(A)\cdot B}$ to a weakly finite element of Set^A, by means of $\mathrm{App}^{A,B}$, then the result is finite. Assume that

$$F = \sum_{d,b} C_{d,b} \cdot (d, b), \qquad u = \sum_a u_a \cdot a.$$

Since $\boldsymbol{F}$ is finite, almost all components F_b of the functor F encoded by $\boldsymbol{F}$ are zero, and this shows that the value $F(\boldsymbol{u})$ is a finite sum. So it suffices to show the values $F_b(\boldsymbol{u})$ are finite and we shall be done.

$F_b(\boldsymbol{u}) = \sum_d C_{d,b} \cdot \mathrm{SET}^A(\boldsymbol{d}, \boldsymbol{u})$ is a finite sum, with coefficients $C_{d,b}$ finite, and the sets $\mathrm{SET}^A(\boldsymbol{d}, \boldsymbol{u})$ are finite as well, since they are monomials $u_1^{n_1} \cdots u_k^{n_k}$ and the u_i's are all finite.

(ii) This means that, if the vector $\boldsymbol{F}$ of $\mathrm{Set}^{\mathrm{Int}(A) \cdot B}$ sends (via the application functor $\mathrm{App}^{A,B}$) finite objects of Set^A into weakly finite objects of Set^B, then $\boldsymbol{F}$ is weakly finite. Now from the expression $F_b(\boldsymbol{u}) = \sum_d C_{d,b} \cdot \mathrm{SET}^A(\boldsymbol{d}, \boldsymbol{u})$ it is plain that $F_b(\boldsymbol{d})$ has a greater cardinality than $C_{d,b}$. But $\boldsymbol{d}$ is finite, so $F_b(\boldsymbol{d})$ is a finite set, and we conclude that $C_{d,b}$ is finite. □

6.3. Theorem. *If t is a closed normalizable λ-term, then t^* is weakly finite.*

Proof. It is enough to prove the result when t is normal. We establish an intermediate result for the case of terms with variables: Let $t[x_1, \ldots, x_n]$ be a normal term, and let $\boldsymbol{d}_1, \ldots, \boldsymbol{d}_n$ be finite objects of SET^A; then

(a) if t does not begin with a λ, then $t^*[\boldsymbol{d}_1, \ldots, \boldsymbol{d}_n]$ is finite;

(b) in general, $t^*[\boldsymbol{d}_1, \ldots, \boldsymbol{d}_n]$ is weakly finite.

The proof is by induction on $t[x_1, \ldots, x_n]$.

(i) If $t[x_1, \ldots, x_n]$ is x_i, then condition (a) is fulfilled.

(ii) If $t[x_1, \ldots, x_n]$ is $u[x_1, \ldots, x_n]\,(v[x_1, \ldots, x_n])$ and u does not begin with a λ, then $u^*[\boldsymbol{d}_1, \ldots, \boldsymbol{d}_n]$ is finite, while $v^*[\boldsymbol{d}_1, \ldots, \boldsymbol{d}_n]$ is weakly finite. Then by 5.2(i), $t^*[\boldsymbol{d}_1, \ldots, \boldsymbol{d}_n]$ will be finite.

(iii) If $t[x_1, \ldots, x_n]$ is $\lambda y\,.\,u[x_1, \ldots, x_n, y]$, given $\boldsymbol{d}_1, \ldots, \boldsymbol{d}_n, \boldsymbol{e}$ all finite, then $u^*[\boldsymbol{d}_1, \ldots, \boldsymbol{d}_n, \boldsymbol{e}]$ is weakly finite, so by 5.2(ii) the coefficients of $u^*[\boldsymbol{d}_1, \ldots, \boldsymbol{d}_n, \cdot]$ are finite, i.e., $t^*[\boldsymbol{d}_1, \ldots, \boldsymbol{d}_n]$ is weakly finite. □

6.4 Remarks. (i) 6.3 holds without any hypothesis on the pair (A, q).

(ii) When (A, q) is regular, then the non-normalizable term $(\lambda x\,.\,x(x))(\lambda x\,.\,x(x)) = Y_{\mathrm{int}}(\lambda x\,.\,x)$ is equal to 0, because $Y_{\mathrm{ext}}(\lambda x\,.\,x)$ is null in SET^A. So non-normalizable terms may have weakly finite interpretations. Y_{ext} has finite coefficients (because the functors corresponding to finite iterations $\boldsymbol{u} \mapsto (\boldsymbol{u} \circ \cdots \circ \boldsymbol{u})(0)$ are weakly finite by 6.3; moreover, in the expression of Y_{ext} as the direct limit of finite iteration functors, observe that all monomials added to the nth approximation when passing from n to $n+1$ are of degree $\geqslant n$), so Y_{int} will be weakly finite in SET^A when (A, q) is regular.

Question 1. Is it true that in any regular pair (A, q), all λ-terms have weakly finite interpretations?

Question 2. Assume that t is a closed λ-term such that t^* is weakly finite in all SET^A's (even when (A, q) is not regular); can we infer from that fact that t is normalizable?

6.5. Definition. If A is a non-empty set and q is a bijection between A and $\mathrm{Int}(A)\cdot A$, we define the classes $!^\alpha\mathrm{Set}^A$ and $?^\alpha\mathrm{Set}^A$, for any ordinal number:

$$!^0\mathrm{Set}^A = !\mathrm{Set}^A, \qquad ?^0\mathrm{Set}^A = ?\mathrm{Set}^A,$$
$$!^{\alpha+1}\mathrm{Set}^A = ?^\alpha\mathrm{Set}^A \to !^\alpha\mathrm{Set}^A, \quad ?^{\alpha+1}\mathrm{Set}^A = !^\alpha\mathrm{Set}^A \to ?^\alpha\mathrm{Set}^A,$$
$$!^\lambda\mathrm{Set}^A = \bigcup_{\alpha<\lambda} !^\alpha\mathrm{Set}^A, \qquad ?^\lambda\mathrm{Set}^A = \bigcap_{\alpha<\lambda} ?^\alpha\mathrm{Set}^A \quad \text{for } \lambda \text{ limit}$$

(for instance $?^\alpha\mathrm{Set}^A \to !^\alpha\mathrm{Set}^A$ consists of all $u \in \mathrm{Set}^A$ which send $?^\alpha\mathrm{Set}^A$ into $!^\alpha\mathrm{Set}^A$).

6.6. Proposition. (i) *If $\alpha \leqslant \beta$, then* $!^\alpha\mathrm{Set}^A \subset !^\beta\mathrm{Set}^A \subset ?^\beta\mathrm{Set}^A \subset ?^\alpha\mathrm{Set}^A$.

(ii) *There exists some ordinal α_0 such that for all $\alpha \geqslant \alpha_0$: $!^\alpha\mathrm{Set}^A = !^{\alpha_0}\mathrm{Set}^A$ and $?^\alpha\mathrm{Set}^A = ?^{\alpha_0}\mathrm{Set}^A$. We put $!^*\mathrm{Set}^A = !^{\alpha_0}\mathrm{Set}^A$ and $?^*\mathrm{Set}^A = ?^{\alpha_0}\mathrm{Set}^A$.*

Proof. (i) Assume that the property holds for α and β; then

$$!^{\alpha+1}\mathrm{Set}^A = ?^\alpha\mathrm{Set}^A \to !^\alpha\mathrm{Set}^A \subset ?^\beta\mathrm{Set}^A \to !^\beta\mathrm{Set}^A = !^{\beta+1}\mathrm{Set}^A;$$

for symmetry reasons, $?^{\beta+1}\mathrm{Set}^A \subset ?^{\alpha+1}\mathrm{Set}^A$. Also

$$!^{\beta+1}\mathrm{Set}^A = ?^\beta\mathrm{Set}^A \to !^\beta\mathrm{Set}^A \subset !^\beta\mathrm{Set}^A \to ?^\beta\mathrm{Set}^A = ?^{\beta+1}\mathrm{Set}^A$$

and this establishes that the property persists for the pair $(\alpha+1, \beta+1)$.

Now, we prove the property for $\beta = \alpha+1$, by induction on α:

– If $\alpha = 0$, this is just Theorem 5.2.

– If $\alpha = \alpha' + 1$, this is a consequence of the induction hypothesis on α'.

– If α is limit, and $u \in !^\alpha\mathrm{Set}^A$, then $u \in !^\gamma\mathrm{Set}^A$ for some $\gamma < \alpha$, and using the induction hypothesis, $u \in !^{\gamma+1}\mathrm{Set}^A$; so if $x \in ?^\alpha\mathrm{Set}^A$, then since $x \in ?^\gamma\mathrm{Set}^A$ by definition of $?^\alpha\mathrm{Set}^A$, it follows that $u(x) \in !^\gamma\mathrm{Set}^A \subset !^\alpha\mathrm{Set}^A$. The general inclusion $!^\alpha\mathrm{Set}^A \subset ?^\alpha\mathrm{Set}^A$ is easily established by induction.

From the case $\beta = \alpha+1$, there is no trouble to obtain the full property.

(ii) $!^\alpha\mathrm{Set}^A$ is a proper class, but in fact, it is completely determined by its subset consisting of objects of $!^\alpha\mathrm{Set}^A$ with integer coefficients: simply because any time u is in $!^\alpha\mathrm{Set}^A$, then all objects isomorphic to u are in $!^\alpha\mathrm{Set}^A$ as well. Then the $!^\alpha\mathrm{Set}^A$'s can be viewed as a family of subsets of the set X of all objects of Set^A with integer coefficients, and since the family is increasing, it must be stationary after some state. Symmetric argument for ?. □

6.7. Remark. (i) If t is a closed normal term, then $t^* \in ?^*\mathrm{Set}^A$: simply because 6.2 holds if we replace ? by ?* and ! by !*.

(ii) The inclusion between $!^*\mathrm{Set}^A$ and $?^*\mathrm{Set}^A$ is strict, with a rather funny proof from the constructive viewpoint:

(1) The equality is impossible: if we have equality, then we can extend the argument of Theorem 6.3 so to get that $t^* \in ?^*\mathrm{Set}^A$ for any closed t. But if t is Y_{int},

it is easy to find a finite d such that all fixed points of d are infinite, and then $t^*(d) \notin ?^*\mathrm{Set}^A$.

(2) So the identity functor does not map $?^*\mathrm{Set}^A$ into $!^*\mathrm{Set}^A$, so the object $(\lambda x \,.\, x)^*$ is in $?^*\mathrm{Set}^A - !^*\mathrm{Set}^A$.

(iii) It is possible to chop Set^A into slices:

- I consists of all objects which are not weakly finite,
- W_α consists of $?^\alpha\mathrm{Set}^A - ?^{\alpha+1}\mathrm{Set}^A$,
- N consists of $?^*\mathrm{Set}^A - !^*\mathrm{Set}^A$,
- S_α consists of $!^{\alpha+1}\mathrm{Set}^A - !^\alpha\mathrm{Set}^A$,
- F consists of $!\mathrm{Set}^A$

(infinite, weak, normal, strong, finite).

Every object of Set^A falls into exactly one of these cases. For instance when (A, q) is regular, Y^*_{int} falls into W_0 (because there are finite objects whose fixed points are all infinite) but $(Y_{\mathrm{int}}(\lambda x \,.\, x))^* = 0$ falls into F. It is expected that no normal t^* falls into some S_α or F, so all normal terms should fall into N. The behaviour of t^* with respect to the slices is rather unknown; it may depend on many things, such as the normality of (A, q). Also which slices W_α and S_α can be reached by a real t^* (for instance all recursive α's?)?

6.8. Remark. What is the hidden reason that makes things work in 6.3? The reason is the relationship of normality (and cut-elimination results) with three-valued semantics. See [2] for instance. If one considers non-deterministic three-valued models (Schütte's 'semi-valuations'), then these models are complete w.r.t. cut-free provability. In the case of implication the 'truth table' of a semi-valuation is as follows;

- if $A \to B$ is true, then either A is false or B is true;
- if $A \to B$ is false, then A is true and B is false.

This is a non-deterministic notion, because for instance from the truth of A and the falsity of B we cannot infer the falsity of $A \to B$ which may be undeterminated.

When A is a formula, $!A$ means "A is true" and $?A$ means "A is not false". In such a model the following rules of inference are valid:

(i) If $!(A \to B)$ and $?A$, then $!B$;

(ii) if $?B$ follows from the hypothesis $!A$, then $?(A \to B)$.

These two properties are the exact analogs of 6.2 (i) and (ii) (recall that according to Curry–Howard, application corresponds to Modus Ponens, while λ-abstraction corresponds to the deduction theorem).

A three-valued model is a structure where the values can be computed in a deterministic way from the values (true, false, undeterminated) of the atomic sentences:

- $A \to B$ is true iff either A is false or B is true;
- $A \to B$ is false iff A is true or B is false.

In such models, we have also the symmetric situations

(iii) if $?(A \to B)$ and $!A$, then $?B$;

(iv) if $!B$ follows from the hypothesis $?A$, then $(A \to B)$.

The replacement of ! and ? by $!^*$ and $?^*$ corresponds therefore to the replacement of a non-deterministic model by a deterministic one.

Let us now turn our attention towards Gödel's $\mathcal{T}$.

6.9. Definition. If σ is a type, we define the class $.\mathrm{Set}^{\sigma} \subset \mathrm{Set}^{\sigma^*}$ of hereditarily finite objects of type σ:

(i) $.\mathrm{Set}^{int} = !\mathrm{Set}^{int^*}$ (the class of all finite objects of type K),

(ii) $.\mathrm{Set}^{\sigma \times \tau} = .\mathrm{Set}^{\sigma} \times .\mathrm{Set}^{\tau}$,

(iii) $.\mathrm{Set}^{\sigma \to \tau} = .\mathrm{Set}^{\sigma} \to .\mathrm{Set}^{\tau}$.

6.10. Proposition. *For each type σ, we have the following inclusions:*

$$!\mathrm{Set}^{\sigma^*} \subset .\mathrm{Set}^{\sigma} \subset ?\mathrm{Set}^{\sigma^*}.$$

Proof. Easy exercise. □

6.11. Theorem. *If t is a closed term of Gödel's $\mathcal{T}$, then t^* is weakly finite.*

Proof. We shall in fact prove that t^* is hereditarily finite: by induction on the term $t[x_1, \ldots, x_n]$ we show that given any $u_1, \ldots, u_n$ hereditarily finite of the appropriate types, then $t^*[u_1, \ldots, u_n]$ is hereditarily finite. The proof is straightforward, and so we shall only consider the case of a recursion: Assume that t and u are closed terms and that t^* and u^* are hereditarily finite; then we show that $(Rtu)^*(v)$ is hereditarily finite for all finite v in Set^K; for this we state a lemma:

6.11.1. Lemma. *Any finite linear combination of hereditarily finite objects of type σ is hereditarily finite.*

Proof. Immediate by induction on σ. □

Then we show that $(Rtu)^*(v)$ is hereditarily finite by induction on the index n of the greatest nonzero coefficient of v: write $v = n \cdot N_0 + u \cdot U_0 + T^*(w)$; then

$$(Rtu)^*(v) = n \cdot t^* + u \cdot u^*((Rtu)^*(w), w).$$

– If v is null, then $(Rtu)^*(v)$ is hereditarily finite by the lemma.

– Otherwise, we can apply the induction hypothesis to w, and $(Rtu)^*(w)$ is therefore hereditarily finite; but then $u^*((Rtu)^*(w), w)$ is hereditarily finite as well, and by the lemma, $n \cdot t^* + u \cdot u^*((Rtu)^*(w), w)$ is hereditarily finite. □

6.12. Remark. If Theorem 6.11 were Π^0_2 (it is Π^0_3), then it would not be provable in Peano arithmetic, because from a function which to any closed t of type *int* associates the sum of the coefficients of t^*, one recovers the evaluation function, which associates to t the integer n such that $t \neq n$; in fact the two functions are the same. But due to the high logical complexity of Theorem 6.11 its provability in PA is open.

7. Normal functors and probabilistic algorithms

We have so far developed our theory, without bothering at all about the consistency of the informations; for instance an input such as $N_0 + U_0$ which says both that it is zero and nonzero is perfectly acceptable. We shall adopt here a very pragmatic attitude: we still agree that all combinations are possible, but simply some of them are less frequent than others, and the problem of consistency of information will be solved by measuring the frequency of possible combinations. This is clearly a probabilistic, measure-theoretic approach.

This approach will be particularly convincing in the case of a probabilistic algorithm: let us recall that a probabilistic algorithm is an algorithm depending on an external probabilistic oracle, typically at some stage the machine asks for an input Y/N and you give this input by tossing a coin. We have already explained (4.2) how the superposition of all the independent possibilities leads to a certain linear combination. In fact, if we call $\boldsymbol{x}$ this superposition, for each particular execution of the program the output will only be a subobject of $\boldsymbol{x}$. So our information on the random algorithm will be a measure on the set of all subobjects of $\boldsymbol{x}$.

The usual operations on algorithms will transport such measures, so the question is not that of the possibility of putting measures here, but mainly of presenting the measure on the set of subobjects of $\boldsymbol{x}$ in such a way that the transport of this measure is effective.

The solution is that we define the measure by only measuring open sets (in the product topology) which correspond to positive information; this is enough to define the measure, but the most important fact is that if we define the measure in this way, then we shall obtain very nice and simple algorithms for transporting it.

7.1. Definition. Assume that $\boldsymbol{x} \in \mathrm{Set}^A$; to $\boldsymbol{x}$ we can associate the set

$$|\boldsymbol{x}| = \sum_{a \in A} x_a = \{(a, i); a \in A \ \&\ i \in x_a\}.$$

We define $\mathbb{P}(\boldsymbol{x})$ to be the power set $\mathbb{P}(|\boldsymbol{x}|)$, $\mathbb{P}_f(\boldsymbol{x})$ to be the set of finite subsets of $|\boldsymbol{x}|$. Any element of $\mathbb{P}(\boldsymbol{x})$ can be identified with a subobject of $\boldsymbol{x}$, i.e., with a family $\sum y_a \cdot \boldsymbol{a}$ where $y_a \subset x_a$ for all a.

Given $e \in \mathbb{P}_f(\boldsymbol{x})$ we can define the sets:

$$O_e = \{y; y \in \mathbb{P}(\boldsymbol{x}) \ \&\ e \subset y\}, \qquad F_e = \{y; y \in \mathbb{P}(\boldsymbol{x}) \ \&\ e \cap y = \emptyset\}.$$

Since $O_d \cap O_e = O_{d \cap e}$ and $F_d \cap F_e = F_{d \cap e}$, it follows that the sets $O_d \cap F_e$ are closed under finite intersections, and it turns out that the usual product topology on $\mathbb{P}(x)$ is exactly the topology where the open sets are arbitrary unions of sets of the form $O_d \cap F_e$.

7.2. Definition. Assume that $x \in \mathrm{Set}^A$; a *consistency measure* on x is an application m from $\mathbb{P}_f(x)$ into $\mathbb{R}$ enjoying the following properties:

(i) $m(\emptyset) = 1$.

(ii) For any $d, e \in \mathbb{P}_f(x)$, such that $d \cap e = \emptyset$, consider the real numbers

$$M(d, e) = \sum_{c \subset e} (-1)^{\mathrm{card}(c)} m(d \cup c)$$

Then $0 \leq M(d, e)$.

7.3. Examples. (i) Let μ be a probability measure on $\mathbb{P}(x)$ and define an application m from $\mathbb{P}_f(x)$ to $\mathbb{R}$ by

$$m(e) = \mu(O_e)$$

Then m is a consistency measure on x:

– Condition (i) holds because $O_\emptyset = \mathbb{P}(x)$.

– It is easy to check that $M(d, e) = \mu(O_d \cap F_e)$, so $M(d, e) \geq 0$.

(ii) If we choose real numbers m_i for $i \in |x|$, such that $0 \leq m_i \leq 1$ it is possible to define a consistency measure m on x by means of the formula $m(d) = \prod_{i \in d} m_i$. The condition (ii) is satisfied because of the equality

$$M(d, e) = m(d) \prod_{i \in e} (1 - m_i).$$

(iii) A particular case is that of $m_i = 0$ or $m_i = 1$ for all i; then it is still possible to use the product formula of (ii) above to extend m into a consistency measure; but moreover, this is the only possible extension:

– If we consider the numbers $M(d, d \cup \{i\})$, we must have $m(d \cup \{i\}) \leq m(d)$, so the function $d \mapsto m(d)$ is decreasing; from this it follows that $m(d)$ must be zero as soon as some m_i with $i \in d$ is zero.

– Assume that $m_i = 1$ for all $i \in d$; then $m(d) = 1$; this can be established by induction on d. The case $\mathrm{card}(d) \leq 1$ is trivial, so assume that the property holds for all d's of cardinality n. If e is of cardinality $n+1$, and $m_i = 1$ for all $i \in e$, let d be a subset of e of cardinality $n-1$; then $e = d \cup \{i, j\}$,

$$M(d, e) = m(d) - m(d \cup \{i\}) - m(d \cup \{j\}) + m(e) \geq 0, \quad \text{so } m(e) = 1.$$

7.4. Theorem. *Let m be a consistency measure on x. Then there is a unique probability measure on $\mathbb{P}(x)$, μ, such that*

$$\mu(O_e) = m(e) \quad \textit{for all } e \in \mathbb{P}_f(x).$$

Proof. The quantities $M(d, e)$ satisfy the equation

(i) $M(d, e) = M(d, e \cup \{i\}) + M(d \cup \{i\}, e) \quad (i \notin d \cup e)$

while the sets $O_d \cap F_e$ satisfy a similar equation

(ii) $O_d \cap F_e = O_d \cap F_{e\cup\{i\}} + O_{d\cup\{i\}} \cap F_e \quad (i \notin d \cup e)$

(here the sum means a disjoint union). Hence from the values $\mu(O_d) = m(d)$ $(= \mu(O_d \cap F_\emptyset))$ it is immediate to obtain

$$\mu(O_d \cap F_e) = M(d, e).$$

Hence the measure of the basic clopen sets is well determined, so there is at most one solution.

The existence of a solution is a simple inverse limit result: consider the index set $K = \mathbb{P}_f(x)$; when $k \in K$, we can introduce a finite subobject x_k of x by the condition $|x| = k$. The set $\mathbb{P}(x_k)$ is finite, so in order to define a measure on it, it suffices to give the measure of the points: if $y \subset |x_k|$, then define $\mu_k(\{y\}) = M(y, k-1)$.

When $k \geqslant k'$, let $f_{kk'}$ be the continuous function from $\mathbb{P}(k')$ to $\mathbb{P}(k)$ defined by: $f_{kk'}(y) = y \cap k$. Then the measure μ_k is the direct image of the measure $\mu_{k'}$ by the function $f_{kk'}: \mu_k = f^*_{kk'}(\mu_{k'})$. (*Proof.* It suffices to consider the case $k' = k + \{i\}$; if $y \subset \mathbb{P}(k)$, then its measure for the image measure $f^*_{kk'}(\mu_{k'})$ is by definition $\mu_{k'}(f^{-1}_{kk'}(\{y\}))$. But $f^{-1}_{kk'}(\{y\}) = \{y, y \cup \{i\}\}$, so $\mu_{k'}(f^{-1}_{kk'}(\{y\})) = M(y, k'-y) + M(y + \{i\}, k - y) = M(y, k-y)$. □)

Hence $(\mathbb{P}(x_k), \mu_k; f_{kk'})$ is an inverse system of positive measures. Since the functions $f_{kk'}$ are surjective, a classical result yields the existence of a measure μ on the inverse limit $\mathbb{P}(x)$ of the system $(\mathbb{P}(x_k), f_{kk'})$, and this measure is such that $f^*_k(\mu) = \mu_k$. (See e.g. Bourbaki, Intégration, Ch. III, § 4, 5.8.) In particular, $\mu_d(\{d\}) = \mu(f_d^{-1}(\{d\})) = \mu(O_d)$, and so $\mu(O_d) = m(d)$. □

To explain how his may be used, we shall give some examples:

7.5. Examples. Let us see what happens with the product of types: given $x \in \mathrm{Set}^A$, $y \in \mathrm{Set}^B$ and consistency measures $m(\cdot)$ and $n(\cdot)$ on x and y, we want to define a consistency measure on what represents the pair (x, y) namely the sum $x + y$ (as usual, A and B are taken disjoint).

(i) $|x + y| = |x| \cup |y|$; if $d \in \mathbb{P}_f(x + y)$, it is possible to define $p(d) = m(d \cap |x|) \cdot n(d \cap |y|)$. This is a consistency measure because the measure it induces on $\mathbb{P}(x + y) = \mathbb{P}(x) \cdot \mathbb{P}(y)$ is just the product of the measures induced by $m(\cdot)$ and $n(\cdot)$.

(ii) In example (i), we have considered that what we do on x is independant on what we do on y, so the common probabilities were products. In many cases x and y are not independant: for instance take $x = y$, $m(\cdot) = n(\cdot)$; we form the sum of two copies x' and x'' of x and we know that these objects are the same. Then it is natural to define $p(\cdot)$ on $x' + x''$ by $p(d_1' + d_2'') = m(d_1 \cap d_2)$. The measure induced by $p(\cdot)$ on the product $\mathbb{P}(x) \cdot \mathbb{P}(x)$ is the measure $\nu(S) = \mu(\{x; (x, x) \in S\})$ where μ is the measure induced by $m(\cdot)$. In that case the measure is supported by the diagonal.

(iii) In both cases the restriction of the consistency measure $p(\cdot)$ to x and y respectively is $m(\cdot)$ and $n(\cdot)$.

7.6. Examples. Let us now turn our attention towards application: the general case is that of the application of a probabilistic function to a probabilistic input. Moreover, the consistency measures on the function and on the arguments may be related, i.r., the consistency measure on the pair may be different from the product consistency measure of 7.5.1. If we call u the (vector encoding the) function, v the argument, this means that we have a consistency measure on $u + v$, say $m(\cdot)$, and we want to define a consistency measure on $\mathrm{App}(u, v)$. So a way of attacking the problem is to find in which way a consistency measure is transported by a normal functor (here the normal functor is the functor $\mathrm{App}^{A,B}$ from $\mathrm{SET}^{\mathrm{Int}(A) \cdot B + A}$ to SET^B).

(i) If F is a normal functor from SET^A to SET^B, $x \in \mathrm{Set}^A$, $m(\cdot)$ a consistency measure on x, and μ the associated measure on $\mathbb{P}(x)$, then we define the consistency measure $F(m(\cdot)) = n(\cdot)$ on $F(x)$ by the condition that the measure ν associated with $n(\cdot)$ enjoys the condition: $\nu = f^*(\mu)$, where the function f from $\mathbb{P}(x)$ to $\mathbb{P}(F(x))$ is defined by $f(S) = \mathrm{rg}(F(\mathrm{inc}_{sx}))$ where S is a subset of $|x|$, s the associated subobject of x and inc_{sx} the inclusion map.

So let $z_1, \ldots, z_k$ be points of $F(x)$; write $z_i = (b_i, w_i)$, and it is possible to write the normal form of w_i w.r.t. F_{b_i} as:

$$w_i = |w_i^*; d_i; f_i; x|_{F_{b_i}}.$$

$f_i \in \mathrm{SET}^A(d_i, x)$, so one can obtain functions $|f_i|$ from $|d_i|$ to $|x|$ and we shall use the notation $\mathrm{rg}(f_i)$ for the range of the function $|f|$.

Then $z_i \subset f(S)$ iff $\mathrm{rg}(f_i) \subset S$, so $\{z_1, \ldots, z_k\} \subset f(S)$ iff $\mathrm{rg}(f_1) \cup \cdots \cup \mathrm{rg}(f_k) \subset S$. This proves that the inverse image under f of the open set $O_{\{z_1,\ldots,z_k\}}$ of $F(x)$ is the open set $O_{\mathrm{rg}(f_1)\cup\cdots\cup\mathrm{rg}(f_k)}$ of x, and so we have obtained the formula:

$$n(\{z_1, \ldots, z_k\}) = m(\mathrm{rg}(f_1) \cup \cdots \cup \mathrm{rg}(f_k)).$$

(ii) The formula obtained is the general form when F is a normal functor, not viewed as a probabilistic algorithm. Now, if we use the application functor, this formula can also be used to give the new consistency measure $n(\cdot)$ given the original consistency measure on the sum of u (encoding F) and x. The formula obtained is simply

$$n(\{z_1, \ldots, z_k\}) = m(\mathrm{rg}(f_1) \cup \cdots \cup \mathrm{rg}(f_k) \cup \{(d_1, w_1^*), \ldots, (d_k, w_k^*)\})$$

(recall that F is encoded by selecting an integral normal form in each equivalence class of normal forms, and we assume that the pairs (d_i, w_1^*) have been selected).

(iii) An important case is when the function and the argument are independant, and the formula becomes

$$n(\{z_1, \ldots, z_k\}) = m(\mathrm{rg}(f_1) \cup \cdots \cup \mathrm{rg}(f_k)) \cdot m'(\{(d_1, w_1^*), \ldots, (d_k, w_k^*)\})$$

where $m(\cdot)$ and $m'(\cdot)$ are the consistency measures equipping x and u respectively.

(iv) The original case (i) is indeed a particular case of (iii): simply put $m'(d) = 1$ for all $d \in \mathbb{P}_f(u)$. This choice, in probabilistic terms corresponds to putting a Dirac measure of 1 at the subset $|u|$.

7.7. Discussion. Let us now see if it is possible to use general measures.

(i) If we drop the requirement that the total measure is 1, then this opens the possibility to have infinitely many antagonistic choices with the same weight. It would be possible in this way to express what is an acceptable input of some kind; we do it in the case of type *int*: what is sure is that the input is denumerable, so we shall work with

$$x = \mathbb{N} \cdot N_0 + \mathbb{N} \cdot U_0 + \mathbb{N} \cdot N_1 + \mathbb{N} \cdot U_1 + \cdots$$

Then we shall define, when $d \in \mathbb{P}_f(x)$, $m(d)$ by: $m(d) = 1$ if the subobject d corresponding to d is isomorphic to a subobject of some $U_0 + U_1 + \cdots + U_{n-1} + N_n$; $m(d) = 0$ otherwise.

This corresponds to putting Dirac measures 1 at every subobject d of x isomorphic to some $S^*(\cdots D^*(0^*)\cdots)$.

(ii) If we drop the requirement that the measures are positive (and why not real), then the formalism still works; however the author confesses that he no longer sees what could be a decent interpretation of such measures. Presumably, this could have something to do with negative information.

Appendix A: Qualitative domains

A.1. Definition. A *qualitative domain* is any set X such that

(i) $\emptyset \in X$.

(ii) If $a \in X$ and $b \subset a$, then $b \in X$.

(iii) X is closed under directed unions.

A.2. Definition. Let A, B be two qualitative domains. Then one defines their product $A \times B$ as follows:

$$A \times B = \{\{0\} \cdot a \cup \{1\} \cdot b; a \in A, b \in B\}.$$

It is immediate to check that the product of two qualitative domains is again a qualitative domain.

A.3. Definition. Let A and B be two qualitative domains. A function f from A to B is said to be *normal* when the following hold:

(i) f is increasing w.r.t. inclusion.

(ii) f is continuous w.r.t. directed unions.

(iii) For all $a, a' \in A$, if $a \cup a' \in A$, then $f(a \cap a') = f(a) \cap f(a')$.

Let f, g be two normal functions from A to B. Then $f \subset g$ means that for all $a, b \in A$ such that $a \subset b$, $f(a) = f(b) \cap g(a)$.

A.4. Theorem. *The set of all normal functions from the qualitative domain A to the qualitative domain B is (isomorphic to) a qualitative domain.*

Proof. 'Isomorphic' means that the bijection transforms the relation $\subset$ between normal functions into the inclusion of the qualitative domain. So let f be a normal function, and let us consider the set [illegible] consisting of all couples (a, z) such that:

(i) $a \in A$ and $z \in f(a)$.

(ii) If $b \subset a$ and $z \in f(b)$, then $b = a$.

First observe that the function $f \mapsto K(f)$ is injective: this is a consequence of the lemma

A.4.1. Lemma. *If $a \in A$, then $f(a) = \{z; \exists a' \subset a\ ((a', z) \in K(f))\}$.*

Proof. Let us call $g(a)$ the right-hand side of the equation; the inclusion $g(a) \subset f(a)$ is immediate from condition (i) of A.3. To prove the converse inclusion, take $z \in f(a)$: then by A.3(ii) one can find $a' \subset a$ finite such that $z \in f(a')$ (since a is the directed union of its finite subsets), and if a' is chosen minimal w.r.t. inclusion, then $(a', z) \in K(f)$, so $z \in g(a)$. $\square$

Then observe that $f \subset g \rightarrow K(f) \subset K(g)$: assume that $f \subset g$ and let $(a, z) \in K(f)$; since the condition $f \subset g$ implies $f(a) \subset g(a)$, it follows that $z \in g(a)$. Now assume that $b \subset a$ and $z \in g(b)$; then $f(b) = f(a) \cap g(b)$ so $z \in f(b)$ and this forces $a = b$: this proves that $(a, z) \in K(g)$.

Conversely let g be a normal function from A to B and let X be any subset of $K(g)$; we shall construct $f \subset g$ such that $K(f) = X$: f is defined by the formula inspired from A.4.1:

$$f(a) = \{z; \exists a' \subset a\ ((a', z) \in X)\}.$$

Then observe that:

(i) $K(f) = X$: assume that a is minimal such that $z \in f(a)$; then it is immediate that $(a, z) \in X$.

(ii) $f \subset g$: if $a \subset b$, then $f(a) \subset f(b) \cap g(a)$ (trivial); conversely assume that $z \in f(b) \cap g(a)$: this means the existence of $b' \subset b$ such that $(b', z) \in X$ and of $a' \subset a$ such that $(a', z) \in K(g)$. But then $a' \cup b' \in A$, so $z \in f(a' \cap b')$ by A.3(iii), and this forces $a' = b' = a' \cap b'$. So $(a', z) \in X$ and $z \in f(a)$.

The qualitative domain we are seeking will be just the set C of all sets $K(f)$, when f varies through all normal functions from A to B. K is an order-preserving bijection between the set of all normal functions from A to B and the set C. But is C a qualitative domain? We have already seen that any subset of an element of C is again an element of C. $\emptyset$ belongs to C because $\emptyset = K(f)$ where f is the normal function $a \mapsto \emptyset$. Finally, it is easily checked that C is closed under directed unions. □

A.5. Theorem. *There is a qualitative domain D which is non-trivial and such that D is isomorphic to the set of all normal functions from D to D.*

Proof. If A and B are qualitative domains, then Theorem A.4 constructs a new qualitative domain $N(A, B)$, corresponding to the set of all normal functions from A to B. By definition, $c \in N(A, B)$ iff c is a set of couples (a, z) where:

(i) a is a finite element of A,

(ii) $\{z\} \in B$, and such that:

(iii) given any $a' \in A$ (it suffices to restrict to the case a' finite), then the set $\{z; \exists a \subset a' ((a, z) \in c)\}$ belongs to B.

(iv) If (a, z), $(b, z) \in c$ and $a \cup b \in A$, then $a = b$.

Qualitative domains can be made into a category: if A and B are qualitative domains, let $|A| = \bigcup A$ etc. Then a morphism from A to B is an injective function f from $|A|$ to $|B|$ such that: for all $a \in |A|$, let $b = \{f(z); z \in a\}$; then $a \in A$ iff $b \in B$.

It is easy to show that usual constructions between qualitative domains such as product, $N(\cdot, \cdot)$, can be made functorial; moreover these functors will usually enjoy nice commutation properties, typically direct limits, pull-backs.

Then it is clear that the functor $D \mapsto N(D, D)$ has a lot of fixed points... We leave the details to the reader. □

A.6. Remarks. (i) From A.5 it is not difficult to find a model for λ-calculus.

(ii) Similar ideas can be used to give a model for Gödel's $\mathcal{T}$; the only thing we need is a qualitative domain for the type *int* (since the product is taken care of by A.2). It can obviously be done in the spirit of Section 4, by ordering the set of all subsets a of $\{N_0, U_1, N_1, U_2, \ldots\}$ such that if $N_i \in a$, then no N_j belongs to a for $j \neq i$ and no U_j belongs to a for $j > i$.

It is not difficult to interpret Gödel's $\mathcal{T}$ in this framework. The gain w.r.t. usual interpretations comes from condition A.3(iii) which has an obvious meaning in terms of stability (see A.8).

A.7. Remark. Qualitative domains can be viewed as a simplification of two different approaches:

(i) In [8] Scott considered domains as follows: X being a set, let S be a set of intuitionistic sequents $a_1, \ldots, a_n \vdash$ and $a_1, \ldots, a_n \vdash b$ with $a_1, \ldots, a_n, b \in X$. The usual rules of sequent calculus (left structural rules and the cut) enable us to define

– Consistent subsets of X,

– saturated subsets of X (under the notion of consequence).

It is always possible to assume that S contains no axiom of the form $a \vdash$. The domain consists of all consistent saturated subsets of X. Then, if A is a domain:

(i) $\emptyset \in A$.

(ii) If $a \in A$ and $b \subset a$ is saturated, then $b \in A$.

(iii) A is closed under directed unions.

So we clearly see the improvement in A.1: there is no longer any saturation condition. The saturation conditions are necessary in Scott's framework because the functions he considers enjoy only A.3(i), (ii), but not A.3(iii). So the adjunction of this condition (inspired from the pull-back preservation property) simplifies the class of domains. Moreover it has obvious consequences in terms of stability (see A.8). Last but not least, what is called 'finite' in Scott domains is not actually finite, only noetherian, whereas in qualitative domains finiteness is finiteness.

(ii) In this paper we have been mainly concerned with a quantitative approach: not only to say when f takes the value . . . at argument . . . , but also how many times it does. Of couse this uneven approach is certainly not always of interest, and this means that we are often more interested to know that some coefficient of a is nonzero than to know its exact value. This therefore suggests a shift of categories: replace SET by a category with only two objects 0 and 1, and only one non-trivial morphism from 0 to 1. Of course, powers of this category can be viewed as sets $\mathbb{P}(A)$, and there is at most one arrow between any two points of $\mathbb{P}(A)$; in fact the existence of the arrow from a to b is just the condition $a \subset b$. In fact, it is not possible to stay with sets $\mathbb{P}(A)$, and it is necessary to restrict to consistent subsets of $\mathbb{P}(A)$, 'consistent' having a meaning depending on the context. This explains the qualitative domains.

The conditions A.3(i), (ii), (iii) respectively correspond to:

(i) the functoriality of f,

(ii) preservation of direct limits,

(iii) preservation of pull-backs; in a qualitative domain viewed as a category, the diagrams of the form

$$\begin{tikzcd} a \cap a' \arrow[r] \arrow[d] & a \arrow[d] \\ a' \arrow[r] & b \end{tikzcd}$$

where $a, a', b \in A$ and $a, a' \subset b$ are cartesian.

The same remark about cartesianity is behind our definition of the relation $\subset$ between normal functions.

A.8. Remark. Consider an algorithm such as the one in 4.4; this algorithm is not *stable*: this means that the data $(0, N_0)$ and $(N_0, 0)$, both subdata of (N_0, N_0), are minimal data leading to the value N_0; but there is no minimum subdatum of (N_0, N_0) leading to this value, since $F(0, 0) = 0$.

(i) If we want to include such algorithms among the ones we study and still keep nice properties, then it seems necessary to consider quantitative settings like the categories SET^A: such algorithms can be represented by functors with a lot of preservation properties.

(ii) In qualitative domains, such functors cannot be represented since we have stability, namely that minimal data are minimum. So the use of qualitative domains will be particularly interesting in the case of the study of stable algorithms.

A.9. Remark. It seems that the probabilistic approach of Section 7 is still working with qualitative domains; however the author confesses that he did not think seriously about the problem.

Appendix B: Sums of types

Let us first recall the syntax of the sum of types, because the subject is not so widely known:

B.1. Definition. If σ and τ are types, then $\sigma + \tau$ is a type. Corresponding to this new scheme, we introduce new schemes for forming objects, in Gödel's T for instance:

(i) If t is of type σ, then $\amalg^1 t$ is of type $\sigma + \tau$;
if u is of type τ, then $\amalg^2 u$ is of type $\sigma + \tau$.

(ii) If v, w, c are of respective types $\rho, \rho, \sigma + \tau$, if x and y are variables of respective types σ and τ, x not free in w and c, and y not free in v and c, then $\oplus xy . vwc$ is of type ρ.

B.2. Definition. We consider a certain number of equations involving the type $\sigma + \tau$; these equations are usually written as reduction rules:

(i) $\oplus xy . v[x]w[y]\amalg^1 t \dashv v[t]$, $\quad\oplus xy . v[x]w[y]\amalg^2 u \dashv w[u]$.

These rules are the essential ones; any interpretation of the sum that would not satisfy these rules must be discarded. But there are additional rules that may be of some value:

(ii) $\oplus xy . \amalg^1 x \amalg^2 y c \dashv c$.

This rule is the formal analog of the 'η-rules' $\lambda x . c(x) \dashv c$ and $(\pi^1 c, \pi^2 c) \dashv c$; this is the 'disjunctive η-rule'.

(iii) Then comes the archipelago of 'commutative rules'; they are justified by the need for a subformula property in the underlying natural deduction system. The general pattern for such rules is the following: if E is an evaluation operation (elimination applied to a principal formula in the natural deduction jargon), then we write

$$E(\oplus xy . vwc) \dashv \oplus xy . E(v)E(w)c.$$

The possible evaluations are:

– When ρ is an implication $\rho' \to \rho'' : E(u) = u(d)$, d a given element of type ρ'. The rule is therefore

$$(\oplus xy \,.\, vwc)(d) \dashv \oplus xy \,.\, v(d)w(d)c.$$

– When ρ is a product $\rho' \times \rho'' : E(u) = \pi^1 u$ or $E(u) = \pi^2 u$. The rules are therefore:

$$\pi^1(\oplus xy \,.\, vwc) \dashv \oplus xy \,.\, \pi^1 v \pi^1 wc, \qquad \pi^2(\oplus xy \,.\, vwc) \dashv \oplus xy \,.\, \pi^2 v \pi^2 wc.$$

– When ρ is a sum $\rho' + \rho'' : E(u) = \oplus zz' \,.\, abu$, where a, b are given terms. The corresponding rule is therefore

$$\oplus zz' \,.\, ab(\oplus xy \,.\, vwc) \dashv \oplus xy \,.\, (\oplus zz' \,.\, abv)(\oplus zz' \,.\, abw)c.$$

If we try to give a model of disjunctive types, then it will be nice if we can fulfill the commutation conditions.

B.3. Definition. Assume that σ^* and τ^* have been defined. Then $(\sigma + \tau)^*$ is by definition the disjoint sum of $\mathrm{Int}(\sigma^*)$ and $\mathrm{Int}(\tau^*)$.

B.4. Definition. The interpretation of the schemes for objects of type $\sigma + \tau$ is as follows (we forget the free variables occurring in the terms, unless it is absolutely necessary to make them appear):

(i) Assume that t is of type σ; then

$$(\iota^1 t)^* = \sum_{d \in \mathrm{Int}(\sigma^*)} t^{*d} \cdot (0, d).$$

Assume that u is of type τ; then

$$(\iota^2 u)^* = \sum_{d \in \mathrm{Int}(\tau^*)} u^{*d} \cdot (1, d).$$

(ii) Assume that $v[x]$, $w[x]$, and c are of respective types ρ, ρ and $\sigma + \tau$, and that x and y are of respective types σ and τ. Then we have already normal functors v^* and w^* together with c^*. We can write

$$v^*[x] = \sum_{d \in \mathrm{Int}(\sigma^*)} x^d \cdot v_d \quad \text{and} \quad w^*[y] = \sum_{d \in \mathrm{Int}(\tau^*)} y^d \cdot w_d.$$

Assume that

$$c^* = \sum_{d \in \mathrm{Int}(\sigma^*)} c'_d \cdot (0, d) + \sum_{d \in \mathrm{Int}(\tau^*)} c''_d \cdot (1, d).$$

Then we define

$$(\oplus xy \,.\, vwc)^* = \sum_{d \in \mathrm{Int}(\sigma^*)} c'_d \cdot v_d + \sum_{d \in \mathrm{Int}(\tau^*)} c''_d \cdot w_d.$$

B.5. Theorem. *All the reduction rules considered in* B.2 *are valid for our interpretation.*

Proof. (i) Consider for instance $(\oplus xy\,.\,vw\varkappa^1 t)^*$; with the notations of B.4(ii) this is equal to

$$\sum_{d\in \mathrm{Int}(\sigma^*)} c'_d\cdot v_d + \sum_{d\in \mathrm{Int}(\tau^*)} c''_d\cdot w_d,$$

so it remains to compute the coefficients c'_d and c''_d. The definition B.4(i) yields $c''_d = 0$, $c'_d = t^{*d}$, hence

$$(\oplus xy\,.\,vw\varkappa^1 t)^* = \sum_{d\in \mathrm{Int}(\sigma^*)} t^{*d}\cdot v_d = v^*[t^*] = (v[t])^*.$$

(ii) If $v[x] = \varkappa^1 x$, then

$$v^*[x] = \sum_{d\in \mathrm{Int}(\sigma^*)} x^d\cdot (0, d)$$

so $v_d = (0, d)$. For similar reasons, if $w[x] = \varkappa^2 x$, it turns out that $w_d = (1, d)$. Then, if we apply the formula B.4(ii), we get

$$(\oplus xy\,.\,\varkappa^1 \times \varkappa^2 yc)^* = \sum_{d\in \mathrm{Int}(\sigma^*)} c'_d\cdot (0, d) + \sum_{d\in \mathrm{Int}(\tau^*)} c''_d\cdot (1, d) = c^*.$$

(iii) First we prove a lemma:

B.5.1. Lemma. *Let F be a linear normal functor from* $\mathrm{SET}^{\rho *}$ *to* $\mathrm{SET}^{\theta *}$. *Then*

$$F((\oplus xy\,.\,vwc)^*) = (\oplus xy\,.\,F(v^*)F(w^*)c)^*.$$

(There is some abuse of notations in the lemma.)

Proof.

$$F((\oplus xy\,.\,vwc)^*) = \sum_{d\in \mathrm{Int}(\sigma^*)} c'_d\cdot F(v_d) + \sum_{d\in \mathrm{Int}(\tau^*)} c''_d\cdot F(w_d).$$

On the other hand

$$F(v^*[x]) = \sum_{d\in \mathrm{Int}(\sigma^*)} x^d\cdot F(v_d) \quad \text{and} \quad F(w^*[y]) = \sum_{d\in \mathrm{Int}(\tau^*)} y^d\cdot F(w_d).$$

Hence

$$(\oplus xy\,.\,F(v^*)F(w^*)c)^* = \sum_{d\in \mathrm{Int}(\sigma^*)} c'_d\cdot F(v_d) + \sum_{d\in \mathrm{Int}(\tau^*)} c''_d\cdot F(w_d). \qquad \square$$

Now it suffices to remark that the evaluations that are used in the commutative rules are linear:

$$E(u) = u(d), \quad E(u) = \pi^1(u), \quad E(u) = \pi^2(u), \quad E(u) = \oplus zz'\,.\,abu$$

are all represented by functors $F(u)$ linear in u. (The only purpose of replacing

σ^* by $\mathrm{Int}(\sigma^*)$ is that now the function $u \mapsto d(u)$ which is usually not at all linear, can be represented by a linear functor; the introduction of $\mathrm{Int}(\sigma^*)$ is reminiscent of the use of the tensor algebra in linear algebra). $\square$

B.6. Remarks. (i) In the framework of qualitative domains, our construction can be adapted as follows: given two q.d.'s A and B, their sum C is defined by: $c \in C$ iff

(1) All elements of c are of the form $(0, a)$ with a a finite element of A or $(1, b)$ with b a finite element of B.

(2) The sets $\bigcup\{a; (0, a) \in c\}$ and $\bigcup\{b; (1, b) \in c\}$ belong respectively to A and B. Moreover one of these sets is empty.

Then it is possible to define, when $a \in A$, $\imath^1 a$ by:

$$\imath^1 a = \{(0, a'); a' \text{ finite subset of } a\}$$

and similarly when $b \in B$,

$$\imath^2 b = \{(1, b'); b' \text{ finite subset of } b\}.$$

Now if f, g are normal functions from respectively A and B to D, then one can define a function h from C to D as follows:

$$h(\{0, a\}) = \{i \in f(a); i \notin f(a') \text{ for all } a' \subsetneq a\},$$
$$h(\{1, b\}) = \{i \in g(b); i \notin g(b') \text{ for all } b' \subsetneq b\}$$

and in general $h(c) = \bigcup\{(h(\{t\}); t \in c)\}$.

Using this construction, it is easy to interpret $\oplus xy\,.\,vwc$. All the reduction rules are correctly interpreted: this is because the evaluation functions are interpreted by normal functions enjoying the analog of linearity:

$$e(a) = \{e(\{t\}); t \in a\}.$$

(ii) The interpretation of the sum is by no means a sum. However since it interprets all reasonable equations that can be connected with a sum, this makes it quite acceptable.

(iii) The use of $\mathrm{Int}(A)$ is quite interesting and perhaps should be extended to other situations; for instance if one wants to make the recursor linear in the variable of type K, then it suffices to replace K by $\mathrm{Int}(K)$ etc.

(iv) A particular case of sum is *bool*, i.e, the sum of two empty types. The interpretation is as follows:

$$bool^* = \{T, F\}.$$

The instructions T, F and *If* are interpreted as follows

$$T^* = T, \qquad F^* = F,$$
$$If^*(\alpha \cdot T + \beta \cdot F, a, b) = \alpha \cdot a + \beta \cdot b.$$

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